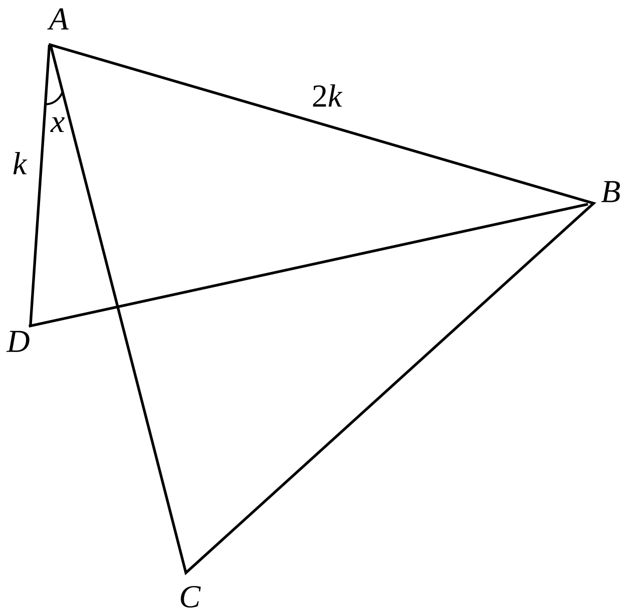
1 **(i)** Solve algebraically the inequality $\dfrac{x-1}{x-2} \leq \dfrac{x-2}{x-1}$. [3]

(ii) Given that $\mathrm{f}\left(x\right)=\dfrac{x-1}{x-2}$, state the value of a such that $\mathrm{f}\left(x-a\right)=\dfrac{2x-3}{2x-5}$. [1]

(iii) Hence solve $\dfrac{2x-3}{2x-5} \leq \dfrac{2x-5}{2x-3}$. [2]

2 In the figure below, *ABC* is an equilateral triangle of length 2*k* units while *ABD* is a triangle where *AD* is of length *k* units and $\measuredangle CAD$ is *x* radians.



(i) Show that $BD^2 = k^2\left(5-2\cos x+2\sqrt{3}\sin x\right)$. [2]

(ii) Given that *x* is sufficiently small such that x^3 and higher powers of *x* may be neglected, find an approximation in *x* for the length *BD*, leaving your answers in exact form and in terms of *k*. [4]

3 It is given that $u_r = \dfrac{1}{r!}$, $r \in \mathbb{Z}^+$.

(i) Show that $u_r - 2u_{r+1} + u_{r+2} = \dfrac{r^2+r-1}{(r+2)!}$. [1]

(ii) Hence find $\displaystyle\sum_{r=1}^{n} \frac{r^2+r-1}{(r+2)!}$ in terms of n and determine the value of $\displaystyle\sum_{r=1}^{\infty} \frac{r^2+r-1}{(r+2)!}$. [3]

(iii) Using an expansion from the List of Formulae MF26 and the answer in part **(ii)**, find the exact value of $\displaystyle\sum_{r=1}^{\infty}\left(u_r - \frac{r^2+r-1}{(r+2)!}\right)$. [2]

4 A curve C has equation $x^2 - 3y^2 = 3$.

(i) Sketch C, giving the equations of asymptote(s) and axial intercept(s) in exact form. [3]

(ii) A complex number is given by $z = x + \mathrm{i}y$, where x and y are real.

Given that $\mathrm{Re}(z)$ is positive, and that x and y satisfy the equation $x^2 - 3y^2 = 3$, state the behaviour of the argument of z when $\mathrm{Re}(z) \to \infty$. [3]

5 A function is defined as $\mathrm{f}(x) = \dfrac{x^3}{x^2+k}$ where k is a constant.

(a) If $k > 0$, show that f is an increasing function. [2]

(b) If $k = -3$,

(i) sketch the graph of $y = \mathrm{f}(x)$, indicating clearly the coordinates of any axial intercept(s), turning point(s) and equation(s) of asymptote(s). [3]

(ii) A curve C has equation $\dfrac{(x-3)^2}{9} + \dfrac{y^2}{a} = 1$, where a is a positive constant. Determine the range of values of a such that C intersects $y = \mathrm{f}(x)$ at more than 3 points. [2]

6 The function f is defined by

$$\mathrm{f} : x \mapsto \begin{cases} 5 + \ln x & \text{for } 0 < x < 1, \\ -x^2 + 6x & \text{for } 1 \le x \le a, \end{cases}$$

where a is a constant.

(i) Given that f has an inverse, determine the range of values of a. [2]

(ii) Given that a satisfies the range of values found in part **(i),** sketch the graph of f, indicating clearly the coordinates of the end point(s) in terms of a. [2]

(iii) Find f^{-1} in similar form. [3]

7 **(i)** Find $\int w^2 \tan^{-1} w \, \mathrm{d}w$. [4]

(ii) The curve C is given by the equation $y = (3x-1)\sqrt{\tan^{-1}(3x-1)}$, where $x \ge \frac{1}{3}$.

The region R is bounded by C, the axes and the line $y = \frac{\sqrt{\pi}}{2}$. Using your result in part **(i)**, find the exact volume generated when R is rotated 2π radians about the x-axis. [4]

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8 **Do not use a graphing calculator for this question.**

(i) The complex number $a+\mathrm{i}$ is a root of the equation $z^2-2\sqrt{2}z+b=0$ where a and b are positive real constants. Find the exact values of a and b. [3]

It is given that $\mathrm{f}(z)=z^6-3z^4+11z^2-9$.

(ii) Show that $\mathrm{f}(z)=\mathrm{f}(-z)$. [1]

(iii) It is given that $a+\mathrm{i}$ is a root of the equation $\mathrm{f}(z)=0$. Using the results found in parts **(i)** and **(ii)**, show that $\mathrm{f}(z)$ can be factorised into 3 quadratic factors with real coefficients, leaving your answer in exact form. [4]

9 A curve C is given by the equation

$$\mathrm{e}^{x^2}y=\ln x^2,$$

where $x\neq 0$.

(i) Show that $2x\ln x^2+\mathrm{e}^{x^2}\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2}{x}$. [2]

(ii) Find the equation of the tangent to C at the point P where $x=1$. Given that the tangent at P meets C again at the point Q, find the coordinates of Q, leaving your answer correct to 5 significant figures. [4]

(iii) Determine the acute angle, in degrees, between the tangents to C at the points P and Q. [3]

10 With reference to the origin O, the position vectors of the points A and B are given as $\mathbf{a}$ and $\mathbf{b}$ respectively. It is given that the area of triangle OAB is 16 units^2 and $|\mathbf{b}|=5$ units.

(i) Find the exact value of $|\mathbf{a}\times\mathbf{d}|$, where $\mathbf{d}$ is a unit vector of $\mathbf{b}$. Give the geometrical interpretation of $|\mathbf{a}\times\mathbf{d}|$ in relation to the triangle OAB. [3]

It is also given that $|\mathbf{a}|=8$ and $\measuredangle AOB$ is obtuse.

(ii) Find the value of $\mathbf{a}\bullet\mathbf{b}$. [3]

(iii) The point C divides AB in the ratio $\mu:1-\mu$, where $0<\mu<1$, and $|\mathbf{c}|=\dfrac{\sqrt{401}}{7}$ units. By considering a suitable scalar product, find the vector(s) $\overrightarrow{OC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [4]

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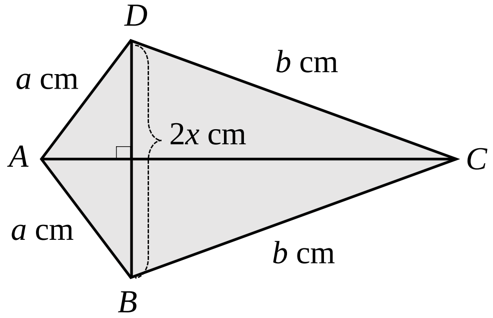
11 Sam wishes to build a kite made with two wooden sticks placed perpendicular to each other, forming the diagonals of the kite AC and BD. BD is given to be $2x$ cm. The perimeter of the kite is to be made from four plastic straws, two of which have fixed lengths a cm each and the other two have fixed lengths b cm each as shown in the diagram below.



(a) **(i)** Show that the surface area of one side of the kite, K cm^2, is given by

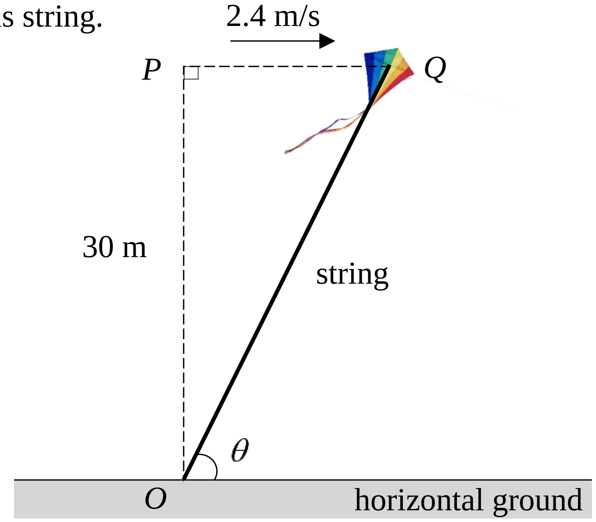
$$K = x\sqrt{a^2 - x^2} + x\sqrt{b^2 - x^2}$$ [2]

(ii) The surface area of a kite is one of the key factors that affect the aerodynamic forces on the flight of kites. To catch as much wind as possible, Sam wishes to maximise the surface area of his kite. Using differentiation, find the value of x in terms of a and b, such that K is maximised. You do not need to show that K is maximum. [4]

(iii) Hence find the length of the wooden stick AC in terms of a and b when K is maximum. [2]

(iv) Using the result in part **(iii)**, deduce the angle between the two straws, AB and BC, when K is maximised. [1]

(b) Sam tests his kite at an open area, and he wants to fly his kite at a fixed vertical height of 30 m above the horizontal ground as shown in the diagram below. The wind blows the kite horizontally from point P to Q with a speed of 2.4 m/s. The string attached to the kite is temporarily tied to the point O on the horizontal ground. Find the rate at which the angle made between the string and the horizontal, θ measured in radians, is decreasing at the instant when Sam lets out $10\sqrt{10}$ m of his string. [4]



12 A cargo drone is used to unload First Aid kit at an accident location in a remote mountain area. The First Aid kit is unloaded vertically through air. The speed of the kit, v ms^{-1}, is the rate of change of distance, x m, of the kit measured vertically away from the drone with respect to time, t seconds.

(i) Write down a differential equation relating v, x and t. [1]

The motion of the First Aid kit is modelled by the differential equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \alpha\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = 10, \qquad \text{(A)}$$

where α is a constant.

(ii) By using the result in part **(i)**, show that the differential equation (A) can be expressed as

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 10 - \alpha v^2.$$ [1]

It is given that when $t = 0$, $v = 0$. It is also given that $\frac{\mathrm{d}v}{\mathrm{d}t} = 4.375$ when $v = 1.5$.

(iii) Find the value of α. By solving the differential equation in part **(ii)**, show that

$$v = \frac{k - k\mathrm{e}^{-10t}}{m + \mathrm{e}^{-10t}},$$

where k and m are constants to be determined. [4]

(iv) Sketch the graph of v against t and describe the behaviour of $\frac{\mathrm{d}x}{\mathrm{d}t}$ and $\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$ in the long run. [4]

(v) Show that the area under the graph in part **(iv)** bounded by the lines $t = 0$ and $t = T$ can be expressed as $\frac{2}{5}\ln\left(\frac{\mathrm{e}^{\beta T} + \mathrm{e}^{-\beta T}}{2}\right)$, where β is a positive integer to be determined. [3]

(vi) What can be said about $\frac{2}{5}\ln\left(\frac{\mathrm{e}^{\beta T} + \mathrm{e}^{-\beta T}}{2}\right)$ in the context of this question? [1]

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