TEMASEK JUNIOR COLLEGE
2020 JC 2 H2 MATHEMATICS (9758)
2020 JC 2 Prelim Exam H2 Mathematics Paper 2

Section A: Pure Mathematics [40 marks]

1 A sequence of numbers u_1, u_2, u_3, … has a sum S_n where $S_n = \sum_{r=1}^{n} u_r$.

(i) It is known that the sum of the first three terms of the sequence is 17 and $\sum_{n=1}^{3} S_n = 30$.

Furthermore, the third term of the sequence is twice the first term of the sequence. Find the first three terms of the sequence. [3]

(ii) A constant k is subtracted from each term of the sequence so that the first three terms of the modified sequence are now three consecutive terms of a geometric progression with a common ratio greater than 1.
Find the value of k and the common ratio of the geometric progression. [3]

2 The functions f and g are defined by

$$\mathrm{f} : x \mapsto 5-(x-2)^2, x \in \mathbb{R},\ x \le 3,$$
$$\mathrm{g} : x \mapsto 3-\mathrm{e}^{-2x},\ x \in \mathbb{R},\ x \le k,$$

where k is a constant.

(i) Sketch the graph of $y = \mathrm{f}(x)$, indicating clearly the coordinates of the turning point, the end-point and the points where the curve meets the axes. Explain why f^{-1} does not exist. [3]

(ii) Explain why the composite function fg exists for all real values of k. [2]

(iii) Find the maximum value of k for which fg is one-one. [3]

(iv) For the value of k found in **(iii)**, find the value of x for which $\mathrm{fg}(x) = (\mathrm{fg})^{-1}(x)$. [2]

3 It is given that $\mathrm{f}(x) = \sin\left[\ln(1+x)\right],\ x \in \mathbb{R},\ x > -1.$

(i) Show that $(1+x)^2 \mathrm{f}''(x) + (1+x)\mathrm{f}'(x) + \mathrm{f}(x) = 0$. [3]

(ii) By further differentiation of the result in **(i)**, find the first three non-zero terms of the Maclaurin series for $\mathrm{f}(x)$. [3]

(iii) Verify the correctness of the series found in part **(ii)** using the standard series from the List of Formulae (MF26). [2]

(iv) Use the series in part **(ii)** to approximate the value of $\int_0^2 \mathrm{f}(x)\,\mathrm{d}x$.

Use your calculator to find an accurate value of $\int_0^2 \mathrm{f}(x)\,\mathrm{d}x$. Why is the approximated value not very good? [3]

(v) Using the series obtained in part **(ii)**, deduce the Maclaurin series for $\cos\left[\ln(1+x)\right]$ in ascending powers of x, up to and including the term in x^2. [2]

4 **(a)** Relative to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors.

(i) It is given that $\mathbf{a}\times\mathbf{b}=\mathbf{0}$. Find $\mathbf{a}\,.\,\mathbf{b}$ in terms of $|\mathbf{b}|$. [2]

It is now given that $\mathbf{a}\times\mathbf{b}\neq\mathbf{0}$. Point P is the foot of the perpendicular from A to the line OB and the point Q is the foot of the perpendicular from B to the line OA. It is given that $AP = BQ$.

(ii) Write down the lengths AP and BQ in terms of $\mathbf{a}$ and $\mathbf{b}$. [2]

Hence show that $|\mathbf{a}|=|\mathbf{b}|$. [1]

(iii) Given also that the angle between $\mathbf{a}$ and $\mathbf{a}-\mathbf{b}$ is ϕ radians, show that $\mathbf{a}.\mathbf{b}=-|\mathbf{a}|^2\cos 2\phi$. [2]

(b) The pyramid $ORST$ has a triangular base ORS and height OT. The position vectors of R and S are $-4\mathbf{i}+\mathbf{j}+3\mathbf{k}$ and $5\mathbf{i}+\mathbf{j}$ respectively. Find the possible coordinates of T if the volume of the pyramid $ORST$ is 35 units^3. [5]

[*Volume of pyramid* $=\dfrac{1}{3}\times$ *base area* $\times$ *height*]

Section B: Probability and Statistics [60 marks]

5 In a chicken farm, eggs are packed into boxes of 12. In the transportation of the eggs from the farm to a supermarket, it is found that on average, k eggs out of 10 eggs are broken on arrival at the supermarket. A box of eggs is randomly chosen for inspection. Find, in terms of k,

(i) the expected number of eggs that are broken in the box, [1]

(ii) the probability that the 8$^{\text{th}}$ egg inspected is the 2$^{\text{nd}}$ egg that is broken. [2]

It is given that $k = 0.8$.

(iii) Find the probability that there are not more than 2 broken eggs in the box. [2]

(iv) If n boxes are randomly selected for inspection, find the least value of n such that the probability that at most 3 boxes will have at least 3 broken eggs is less than 0.9.[3]

6 Javier and Kelvin played a game with a pack of ten cards numbered 1, 2, 3, …, 10. They take turns at drawing two cards each from the pack at random, one card at a time without replacement, and the two cards drawn are replaced before the next player draws. The player who first gets two cards whose sum is 10 wins the game. If he does not win, he replaces the two cards and the other player then takes his turn. The game carries on until a winner is decided.

On their first game, Javier gets to draw first.

(i) Show that the probability that Javier wins the game on his first draw is $\dfrac{4}{45}$. [2]

(ii) Find the probability that Kelvin wins the game. [3]

On their second game, they toss a biased coin to decide on the player to draw first. Using this biased coin, Kelvin is twice as likely as Javier to draw first.

(iii) Find the probability that Javier wins the game. [3]

7 The diagram shows a 3×3 grid of 9 cells, numbered 1 to 9.

1	2	3
4	5	6
7	8	9

The nine letters T, T, T, T, J, C, C, C, C are to be arranged in the grid, with each cell occupied by one letter. Find the number of ways to arrange the nine letters if

(i) there are no other restrictions, [2]

(ii) the cells at the four corners of the grid must each be occupied by a ‘T’. [1]

Two letters are considered adjacent to each other if they occupy cells that are above, below, to the left or to the right of each other. Find the number of ways to arrange the nine letters if

(iii) all ‘T’s must not be adjacent to another ‘T’ and all ‘C’s must not be adjacent to another ‘C’, [2]

(iv) at least two ‘T’s must be adjacent and at least two ‘C’s must be adjacent. [3]

8 The table gives the research and development (R&D) expenditures, x, and earnings, y, in suitable units for 11 pharmaceutical companies in a particular year.

R&D expenditure, x	8.5	12	6.5	4.5	2	0.5	1.5	6	9	7.5	2.5
Earnings, y	83	147	69	50	43	35	40	64	97	53	45

(i) Draw the scatter diagram for these values, labelling the axes clearly. [2]

(ii) Give a possible reason why one of the data points does not seem to follow the trend. [1]

(iii) Using your answer in part **(i)**, explain whether $y = a + bx$ or $y = c + dx^2$ is a better model for the data. [1]

(iv) Explain how you would verify your choice of model in part **(iii)** by calculating the product moment correlation coefficients. [2]

Hence estimate the earnings of a company with a R&D expenditure of 10 units, showing your calculations clearly. [3]

(v) For the model $y = c + dx^2$, give an interpretation, in context, of the value of d. [1]

9 In a TV sports entertainment competition, Spartan Warrior, each contestant has to complete two obstacle stations X and Y in the shortest time possible. It has been found that the time taken to complete station X is normally distributed with mean μ minutes and variance 1.29 minutes^2, and the time taken to complete station Y is also normally distributed with mean 6.5 minutes and variance 1.06 minutes^2. Assume that each contestant begins station Y immediately after completing station X.

(i) If the probability that a randomly chosen contestant takes more than 22.5 minutes to complete both stations is less than 0.0102, find the range of values that μ can take. State an assumption you made in your calculations. [4]

For the remainder of the question, assume that μ = 12.3.

(ii) Find the probability that the time taken by a randomly chosen contestant to complete station X is greater than twice the time taken to complete station Y. [3]

Assume that a contestant begins immediately after the previous contestant completes both stations.

(iii) Find the maximum number of contestants that can be invited to a recording session if the probability that the total time taken for all contestants to complete both stations is within 7 hours is more than 0.99. [3]

(iv) During the broadcast of each episode, the sports channel will broadcast 3 randomly chosen contestants completing both stations without editing. Find the probability that, in one episode, the time taken by one of the 3 randomly chosen contestants to complete both stations is less than 17 minutes and the time taken by the remaining 2 contestants to complete both stations is more than 20 minutes each. [2]

10 Anand and Charlie decide to do some exercise and have fun at the same time. Before the start of each set of exercise, Anand will place 4 blue balls and 5 green balls, identical apart from their colour, into a bag. Charlie will then randomly select 4 balls from the bag without replacement.

X denotes the number of blue balls chosen by Charlie in one set of exercise.

For each blue ball selected, Anand will perform 5 abdominal crunches.

For each green ball selected, Charlie will perform 3 chin-ups.

(i) Show that $\mathrm{P}(X = 3) = \dfrac{10}{63}$ and tabulate the probability distribution of X. [3]

C denotes the number of chin-ups performed by Charlie in 1 set of exercise.

(ii) Without using a graphing calculator, show that $\mathrm{Var}(X) = \dfrac{50}{81}$ and, hence or otherwise, find $\mathrm{Var}(C)$. [5]

(iii) Anand and Charlie decide to do 3 sets of exercise per day for a period of 30 days. Find the approximate probability that during this period, the total number of abdominal crunches performed by Anand is at least 240 greater than the total number of chin-ups performed by Charlie. [4]

11 It is found that certain kinds of meat lose weight as a result of being cooked. A restaurant chef is prepared to accept up to 10% loss but suspects that the recent consignments have a higher percentage weight loss. She decides to carry out a hypothesis test on a random sample of steaks.

(i) Explain the meaning of 'a random sample' in the context of the question and why the chef should sample a large number of steaks. [2]

(ii) State suitable hypotheses for the test, defining any symbols that you use. [2]

The chef takes a random sample of 40 steaks, and calculate the percentage weight loss of each steak, x (in percent). The mean and variance of the percentage weight loss of the 40 steaks are 10.48% with variance 3.37% respectively.

(iii) Test, at the 5% significance level, to determine whether the sample supports the chef's suspicion. [3]

The chef carries out another test using 100 readings obtained from chefs of other restaurants. The percentage weight loss of each steak, y (in percent), is summarised by:

$$\sum y = 1055.2, \qquad \sum y^2 = k\ .$$

(iv) Find the range of values of k such that the chef's suspicion is not valid at the 5% level of significance, giving your answer correct to 2 decimal places. [5]