ST ANDREW'S JUNIOR COLLEGE

PRELIMINARY EXAMINATION

MATHEMATICS **HIGHER 2** **9758/01**

Wednesday **28 August 2019** **3 hrs**

Candidates answer on the Question Paper.
Additional Materials: List of Formulae (MF26)

NAME:____(_____) **C.G.:** __________

TUTOR'S NAME: ____

SCIENTIFIC / GRAPHIC CALCULATOR MODEL: ______________________

READ THESE INSTRUCTIONS FIRST

Write your name, civics group, index number and calculator models on the cover page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions. Total marks : **100**

Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

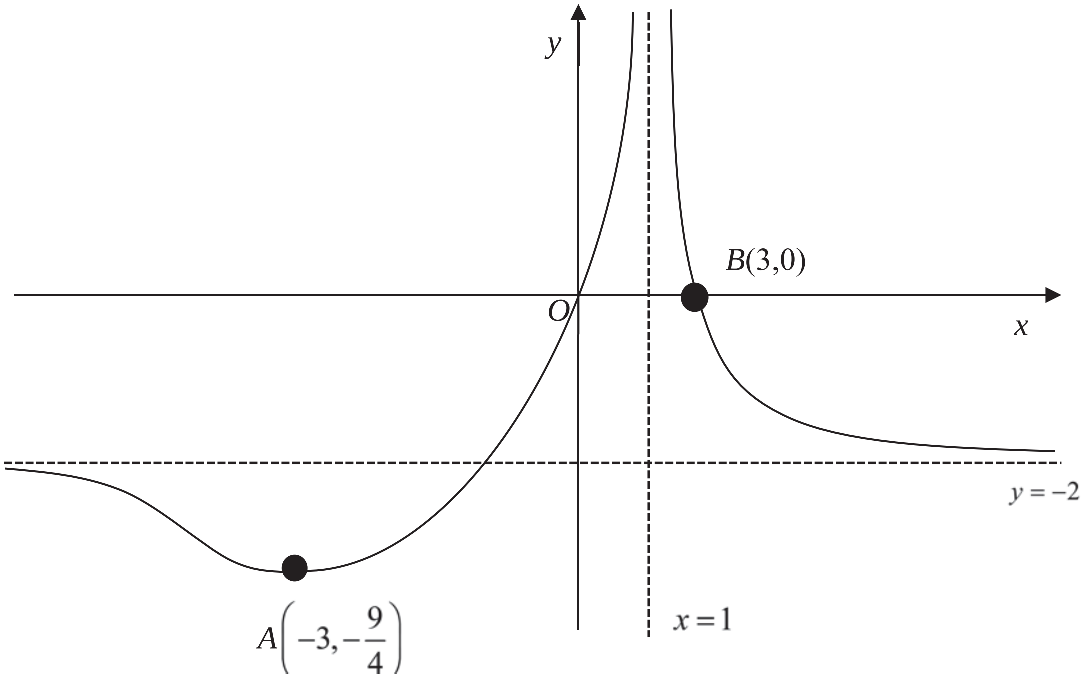
The number of marks is given in brackets [ ] at the end of each question or part question.

Question	1	2	3	4	5	6	7	8	9	10	TOTAL
Marks											
	6	7	10	9	10	10	12	12	12	12	100

This document consists of **27** printed pages and **1** blank page including this page.

[Turn Over]

1 The diagram below shows the graph of $y = 2\mathrm{f}(3-x)$. The graph passes through the origin O, and two other points $A\left(-3,-\dfrac{9}{4}\right)$ and $B(3,0)$. The equations of the vertical and horizontal asymptotes are $x=1$ and $y=-2$ respectively.



(a) State the range of values of k such that the equation $\mathrm{f}(3-x)=k$ has exactly two negative roots. **[1]**

(b) By stating a sequence of two transformations which transforms the graph of $y = 2\mathrm{f}(3-x)$ onto $y=\mathrm{f}(3+x)$, find the coordinates of the minimum point on the graph of $y=\mathrm{f}(3+x)$. Also, write down the equations of the vertical asymptote(s) and horizontal asymptote(s) of $y=\mathrm{f}(3+x)$. **[5]**

2 **(i)** On the same axes, sketch the curves with equations $y=\left|2x^2+6x+4\right|$ and $y=3-4x$, indicating any intercepts with the axes and points of intersection. Hence solve the inequality $3-4x<\left|2x^2+6x+4\right|$. **[4]**

(ii) Find the exact area bounded by the graphs of $y=3-4x$, $y=\left|2x^2+6x+4\right|$, $x=-3$ and $x=-1$. **[3]**

[Turn Over

3 The functions f and g are defined as follows:

$$\mathrm{f}: x \mapsto \frac{x-4}{x-1}\ ,\ x \in \mathbb{R}, x \neq 1$$

$$\mathrm{g}: x \mapsto x^2 + 2x + 2\ ,\ x \in \mathbb{R}, x > -1$$

(i) Show that f has an inverse. **[1]**

(ii) Show that $\mathrm{f} = \mathrm{f}^{-1}$ and hence evaluate $\mathrm{f}^{101}(101)$. **[5]**

(iii) Prove that the composite function fg exists and find its range. **[4]**

4 It is given that $y = \sqrt{\mathrm{e}^{\cos x}}$.

(i) Show that $2\frac{\mathrm{d}y}{\mathrm{d}x} + y\sin x = 0$. Hence find the Maclaurin's expansion of y up to and including the term in x^2. **[4]**

Deduce the series expansion for $\mathrm{e}^{\sin^2\left(\frac{x}{2}\right)}$ up to and including the term in x^2. **[3]**

(ii) Using the series expansion from **(ii)**, estimate the value of $\int_0^{\sqrt{2}} \mathrm{e}^{\sin^2\left(\frac{x}{2}\right)} \mathrm{d}x$ correct to 3 decimal places. **[2]**

5 A curve C is determined by the parametric equations

$$x = at^2, \quad y = 2at, \quad \text{where } a > 0.$$

(i) Sketch C. **[1]**

(ii) Find the equation of the normal at a point P, with non-zero parameter p. **[2]**

Show that the normal at the point P meets C again at another point Q, with parameter q, where $q = -p - \frac{2}{p}$. Hence show that $|PQ|^2 = \frac{16a^2}{p^4}(p^2+1)^3$. **[4]**

(iii) Another point R on C with parameter r, is the point of intersection of C and the circle with diameter PQ. By considering the gradients of PR and QR, show that

$$p^2 - r^2 + 2\left(\frac{r}{p}\right) = 2.$$ **[3]**

[Turn Over

6 (a) (i) Express $p=-1-\sqrt{3}\mathrm{i}$ in exponential form. **[1]**

(ii) Without the use of a calculator, find the two smallest positive whole number values of n for which $\dfrac{(p^*)^n}{\mathrm{i}p}$ is a purely imaginary number. **[4]**

(b) Without the use of a calculator, solve the simultaneous equations

$$z-w+6+7\mathrm{i}=0 \text{ and } 2w-\mathrm{i}z^*-19-3\mathrm{i}=0,$$

giving z and w in the form $x+y\mathrm{i}$ where x and y are real. **[5]**

7 The position vectors, relative to an origin O, at time t in seconds, of the particles P and Q are $(\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + 0\mathbf{k}$ and $\left(\dfrac{3}{2}\cos\left(t+\dfrac{\pi}{4}\right)\right)\mathbf{i}+\left(3\sin\left(t+\dfrac{\pi}{4}\right)\right)\mathbf{j}+\left(\dfrac{3\sqrt{3}}{2}\cos\left(t+\dfrac{\pi}{4}\right)\right)\mathbf{k}$ respectively, where $0\le t\le 2\pi$.

(i) Find $|\overrightarrow{OP}|$ and $|\overrightarrow{OQ}|$. **[2]**

(ii) Find the cartesian equation of the path traced by the point P relative to the origin O and hence give a geometrical description of the motion of P. **[2]**

(iii) Let θ be the angle POQ at time t. By using scalar product, show that

$$\cos\theta=\frac{3\sqrt{2}}{8}-\frac{1}{4}\cos\left(2t+\frac{\pi}{4}\right).$$ **[3]**

(iv) Given that the length of projection of $\overrightarrow{OQ}$ onto $\overrightarrow{OP}$ is $\sqrt{5}$ units, find the acute angle θ and the corresponding values of time t. **[5]**

[Turn Over

8 **(a)** Meredith owns a set of screwdrivers numbered 1 to 17 in decreasing lengths. The lengths of the screwdrivers form a geometric progression. It is given that the total length of the longest 3 screwdrivers is equal to three times the total length of the 5 shortest screwdrivers. It is also given that the total length of all the odd-numbered screwdrivers is 120 cm. Find the total length of all the screwdrivers, giving your answer correct to 2 decimal places. **[4]**

(b) Meredith is building a DIY workbench, and she needs to secure several screws by twisting them with a screwdriver drill. Each time Meredith presses the button on the drill, the screw is rotated clockwise by u_n radians, where *n* is the number of times the button is pressed. Each press rotates the screw more than the previous twist, and on the first press, the screw is rotated by $\frac{2\pi}{3}$ radians. It is given that $\cos u_{n+1} = \frac{1}{2}\cos u_n - \frac{\sqrt{3}}{2}\sin u_n$ and $\sin u_{n+1} = \frac{1}{2}\sin u_n + \frac{\sqrt{3}}{2}\cos u_n$ for all $n \geq 1$.

(i) By considering $\cos\left(u_{n+1} - u_n\right)$ or otherwise, and assuming that the increase in rotation in successive twists is less than π radians, prove that $\{u_n\}$ is an arithmetic progression with common difference $\frac{\pi}{3}$ radians. **[3]**

(ii) Each screw requires at least 25 complete revolutions to ensure that it does not fall out. Find the minimum number of times Meredith has to press the drill button to ensure the screw is fixed in place. **[3]**

(iii) The distance the screw is driven into the workbench on the *n*th press of the drill, d_n, is proportional to the angle of rotation u_n. If the total distance the screw is driven into the workbench after 21 presses is 144mm, find the distance the screw is driven into the workbench on the first press. **[2]**

9 **(i)** By using the substitution $x = 15\sin\theta + 15$, find the $\int_0^{15} \sqrt{15^2 - (x-15)^2}\,dx$ leaving your answer in terms of π . **[5]**

(ii) A sculptor decides to make a stool by carving from a cylindrical block of base radius

30 cm and height 35 cm using a 3D carving machine. The design of the stool based on the piecewise function $g(x)$ where

$$g(x) = \begin{cases} 30 - \frac{2}{3}\sqrt{15^2 - (x-15)^2} & \text{for } 0 \le x \le 15 \\ 30 & \text{for } 15 < x < 35. \end{cases}$$

The figure below shows the 3D image of the stool after the design ran through a 3D machine simulator.

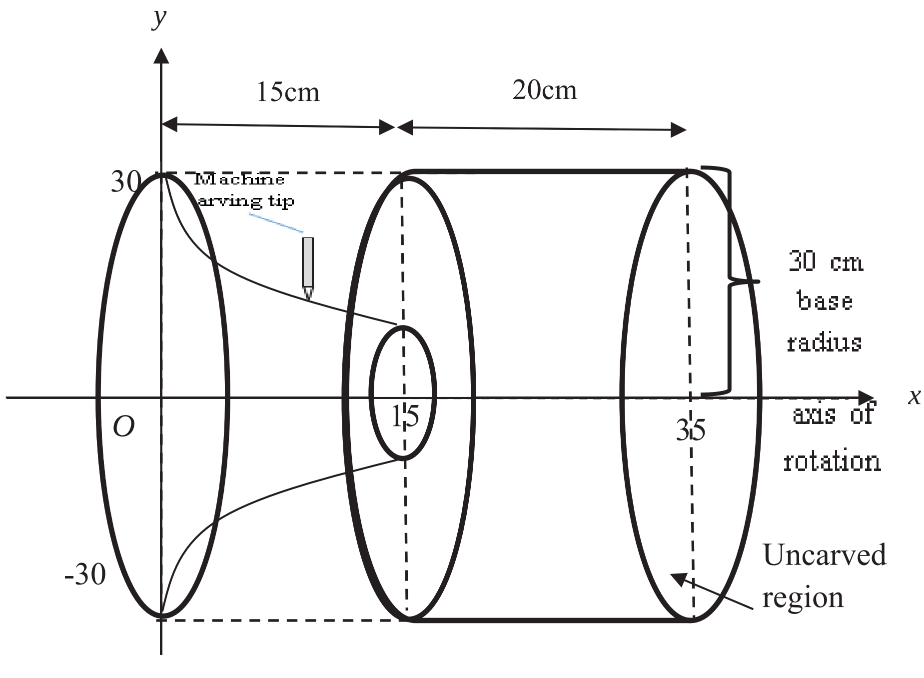


Figure 1: 3D Image of the stool

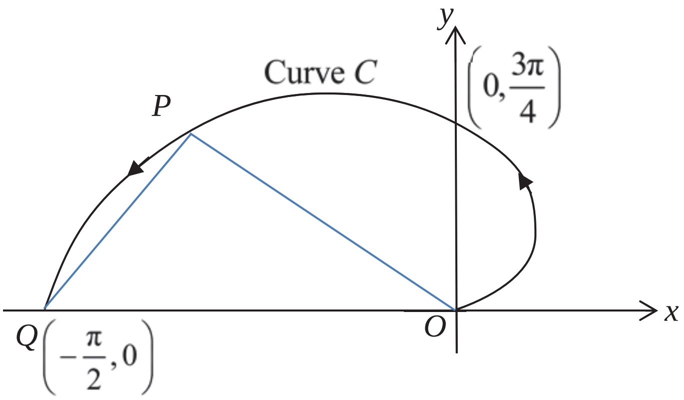
(a) Find the exact area bounded by the curve $y = g(x)$, $x = 15$ and the x-axis and y-axis. **[3]**

(b) The curve defined by the function $y = g(x)$ when rotated 2π radians about the x–axis gives the shape of the stool that the sculptor desires, as shown in Figure 1.
Find the exact volume of the stool. **[4]**

[Turn Over

10 The diagram below shows a curve C with parametric equations given by

$$x = \theta\cos 2\theta, \quad y = 3\theta\sin 2\theta, \qquad \text{for } 0 \le \theta \le \frac{\pi}{2}.$$



The area bounded by curve C and the x-axis is a plot of land which is owned by a farmer Mr Green where he used to grow vegetables. Over the past weeks, vegetables were mysteriously missing and Mr Green decided to install an automated moving surveillance camera which moves along the boundary of the farmland in an anticlockwise direction along the curve C starting from point O and ending at point Q before moving in a clockwise direction along the curve C back to O.

At a particular instant t seconds, the camera is located at a point P with parameter θ on the curve C. You may assume that the camera is at O initially. The camera should be orientated so that the field of view should span from O to Q exactly as shown.

(i) Assuming that the camera is moving at a speed given by $\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0.01$ radians/sec, find the rate of change of the area of the triangle OPQ, A when $\theta = \frac{\pi}{6}$. **[4]**

(ii) Using differentiation, find the value of θ that would maximize A and explain why A is a maximum for that value of θ. Hence find this value of A and the coordinates of the point P corresponding to the location of the camera at that instant. **[5]**

(iii) For the image to be 'balanced', triangle OPQ is isosceles. Find the coordinates of the location where the camera should be. **[3]**

End of Paper