EUNOIA JUNIOR COLLEGE

JC2 Preliminary Examinaton 2020

General Certificate of Education Advanced Level

Higher 2

CANDIDATE NAME	

CLASS		INDEX NO.	

MATHEMATICS **9758/01**

28 August 2020

Paper 1 [100 marks] **3 hours**

Candidates answer on the Question Paper

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name, civics group and question number on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use paper clips, highlighters, glue or correction fluid.
Answer **all** questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **28** printed pages (including this cover page) and **2** blank pages.

For markers' use:												
Q1	**Q2**	**Q3**	**Q4**	**Q5**	**Q6**	**Q7**	**Q8**	**Q9**	**Q10**	**Q11**	**Q12**	**Total**

1 The curve C has equation $x^2 + y^2 = y(x-3)$. Find the coordinates of the points on C at which the tangents are parallel to the y-axis. [4]

2 State the derivative of $\tan x^2$. Hence, or otherwise, find $\int x^3 \sec^2 x^2 \, dx$. [4]

3 [It is given that a sphere of radius r has surface area $4\pi r^2$ and volume $\frac{4}{3}\pi r^3$.]

Air is being blown into a spherical balloon at a constant rate of $12\ \text{cm}^3$ per minute. Initially there is no air in the balloon.

(i) Find the rate at which the radius of the balloon is changing when the radius of the balloon is 5 cm. [2]

(ii) Find the rate at which the surface area of the balloon is changing 10 minutes after air is blown into the balloon. [4]

4 **(i)** On the same axes, sketch the graphs of $y = \dfrac{ab}{|x-1|}$ and $y = b|x-a|$, where a and b are positive constants such that $0 < a < 1 < b$. It is given that the two graphs intersect exactly twice. [3]

(ii) Hence, solve the inequality $|x-a| \le \dfrac{a}{|x-1|}$. [3]

5 You are given that $F = \displaystyle\int \frac{e^x}{e^x + e^{-x}} \, dx$.

(i) Use the substitution $y = e^x$ to find another expression for F. [2]

(ii) By considering $e^x = A(e^x + e^{-x}) + B(e^x - e^{-x})$ where A and B are real constants to be determined, find an expression for F. [3]

(iii) Show algebraically that your answers to parts **(i)** and **(ii)** differ by a constant. [1]

6 The functions f and g are defined by

$$\mathrm{f}: x \mapsto a + \frac{2}{3(x-a)} \quad \text{for } x \in \mathbb{R},\ x \ne a,$$

$$\mathrm{g}: x \mapsto \frac{a}{2} - \left(x - \sqrt{\frac{a}{2}+1}\right)^2 \quad \text{for } x \ge 0$$

where a is a positive constant.

(i) Define, in a similar form, the inverse function f^{-1} and show that $\mathrm{f}^2(x) = x$. [3]

(ii) Hence, find $\mathrm{f}^{2k+1}(2a)$ for positive integer k, giving your answer in terms of a. [2]

(iii) Show that the composite function fg exists and find the range of fg. [3]

7 The sum, S_n, of the first n terms of a sequence $u_1, u_2, u_3, \ldots$ is given by $S_n = a\left(3^n\right) + bn + c$, where a, b and c are constants and $n \geq 1$.

(i) Given that $u_1 = 5, u_2 = 9$ and $u_3 = 33$, find a, b and c. [4]

(ii) Show that $u_{n+1} = A\left(3^n\right) + B$, where A and B are constants to be determined. [3]

(iii) Using your answer in part **(ii)**, find $\sum_{r=2}^{n} u_{r+1}$ in terms of n. (You need not simplify your answer.) [3]

8 **Do not use a calculator in answering this question.**

(a) The complex number z_1 is given by $-1+\mathrm{i}$.

(i) Given that z_1 is a root of the equation $z^2 + az + \left(1-\sqrt{3}\right) + b\mathrm{i} = 0$, find the values of the real numbers a and b. [3]

(ii) Using these values of a and b, find the second root of this equation in exact form. [2]

(b) The complex numbers w_1 and w_2 are given by $2-2\mathrm{i}$ and $-\sqrt{3}+\mathrm{i}$ respectively.

(i) Find the modulus and argument of $w_1 w_2$ in exact form. [3]

(ii) Hence, or otherwise, show that $\cos\frac{7}{12}\pi = \frac{1-\sqrt{3}}{2\sqrt{2}}$ [2]

9 **(a)** **(i)** Given x and y are related by the differential equation $x^2\frac{\mathrm{d}y}{\mathrm{d}x} + xy = k$ for $k \in \mathbb{R}$, show that $y = \frac{k\left(\ln x + \alpha\right)}{x}$ is a solution of the differential equation where α is an arbitrary constant. [2]

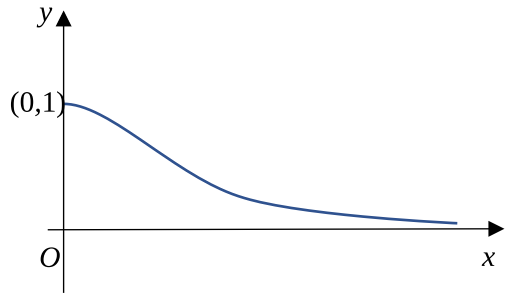
(ii) Hence, show that $\left(\mathrm{e}^{1-\alpha}, k\mathrm{e}^{\alpha-1}\right)$ is a stationary point of the curve $y = \frac{k\left(\ln x + \alpha\right)}{x}$. [2]

(b) It is given that x and y are related by the differential equation $y\frac{\mathrm{d}y}{\mathrm{d}x} + x = \sqrt{x^2 + y^2}$ and that $y = 0$ when $x = -2$.

(i) By substituting $v = x^2 + y^2$, show that the differential equation can be written as $\frac{\mathrm{d}v}{\mathrm{d}x} = 2\sqrt{v}$. [2]

(ii) Find v in terms of x and hence show that $y^2 = \mathrm{f}\left(x\right)$ where $\mathrm{f}\left(x\right)$ is to be determined. [4]

10 The diagram shows the graph of $y = \frac{1}{x^2+1}$ when $x > 0$.



(i) Evaluate $\int_k^{k+1} \frac{1}{x^2+1}\,\mathrm{d}x$ for $k > 0$, leaving your answer in terms of k. [2]

(ii) By considering appropriate rectangles on the interval $[k, k+1]$ for the curve $y = \frac{1}{x^2+1}$, show that

$$\frac{1}{(k+1)^2+1} < \tan^{-1}(k+1) - \tan^{-1} k < \frac{1}{k^2+1} \quad \text{for } k \in \mathbb{Z}^+.$$ [2]

(iii) Use the identity $\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}$ to show that

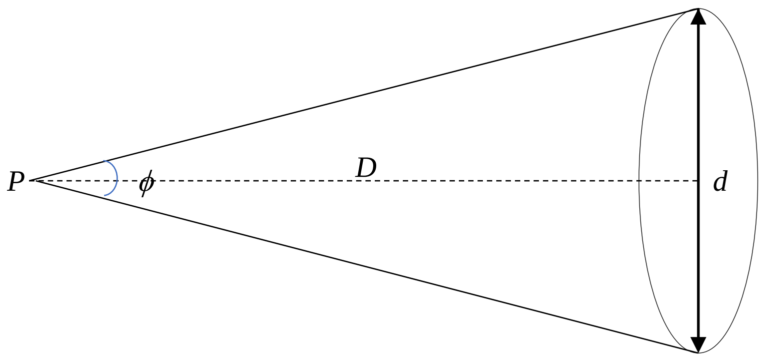
$$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\frac{x-y}{1+xy}, \text{ where } x > y > 0.$$ [2]

(iv) By considering parts **(ii)** and **(iii)**, prove by the method of differences that

$$\sum_{k=1}^{n} \frac{1}{(k+1)^2+1} < \tan^{-1}\left(\frac{n}{n+2}\right) < \sum_{k=1}^{n} \frac{1}{k^2+1}.$$ [4]

11 (a) The angular diameter of an object is the angle the object makes (subtends) as seen by an observer.

As shown in the diagram below, ϕ denotes the angular diameter (measured in radians) of a circle whose plane is perpendicular to the line between the point of view (point P) and the centre of said circle. D denotes the distance from point P to the centre of the circle and d denotes the diameter of the circle.

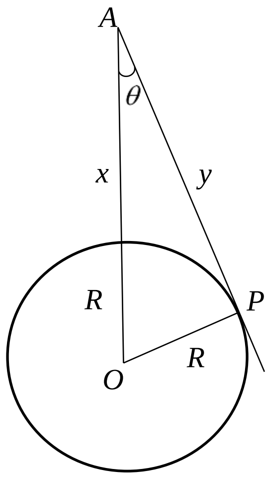


(i) Show that, if ϕ is sufficiently small, $\phi D = d$. [2]

The equation in **(i)** is often used in astronomy to estimate the diameters of stars from the angular diameter, assuming their shapes to be approximately circular.

(ii) If the angular diameter of a star is measured to be 0.00873 rad and the distance of the star from the earth is 9.46×10^{12} km, estimate the diameter of the star. [1]

(b) An astronaut A is at a large distance x km from the surface of the earth. The radius of the earth is assumed to be a constant R km. The furthest point on the earth's surface that the astronaut can see is a point P such that $AP = y$ km and the angle $OAP = \theta$, where O is the centre of the earth (see diagram).

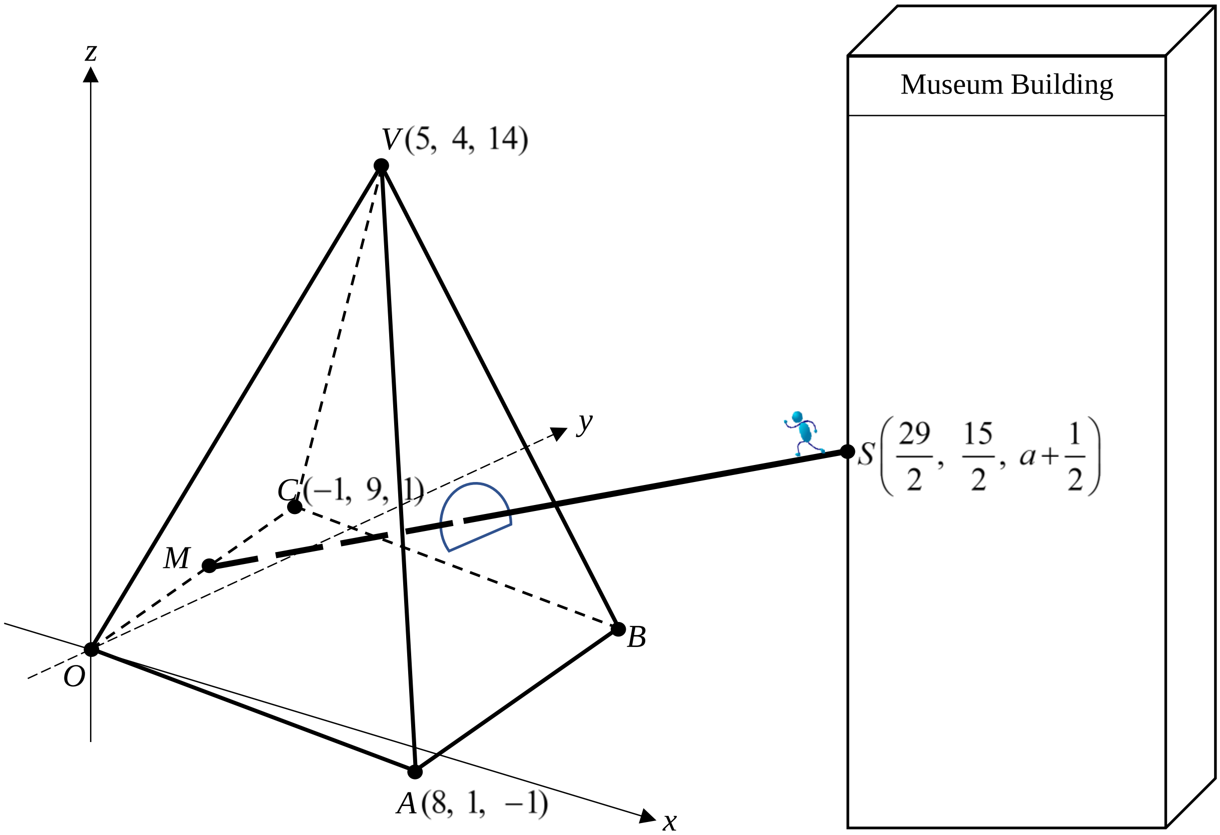


(i) Show that $y = x\left(1+\dfrac{2R}{x}\right)^{\frac{1}{2}}$. [3]

(ii) It is given that R is small compared to x. Show that, if $\alpha = \dfrac{R}{x}$, $\tan\theta \approx \alpha - \alpha^2 + 1.5\alpha^3$. [4]

(iii) It is also given that $\theta = 0.0345$ rad and the astronaut A is 180,000 km from the surface of the earth, find α and hence estimate the radius of the earth. Leave your answer to the nearest km. [3]

12 One of the highlights of the grand opening of the Grand Egyptian Museum in Cairo is a tightrope walking contest. For this contest, as shown in the diagram, a glass pyramid is constructed beside the museum building, with a rectangular base *OABC* and vertex *V*. Points $(x,\ y,\ z)$ are defined relative to $O(0,\ 0,\ 0)$, where units are metres. As the ground is uneven, the pyramid is tilted slightly with *A*, *C* and *V* at $(8,\ 1,\ -1)$, $(-1,\ 9,\ 1)$ and $(5,\ 4,\ 14)$ respectively.



(i) Find a cartesian equation of the plane containing the triangular face *ABV*. [2]

The contestant walks on rope R_1 which is firmly secured to the starting point $S\left(\frac{29}{2},\ \frac{15}{2},\ a+\frac{1}{2}\right)$ on the museum building such that $a \geq 0$. The taut rope R_1 penetrates through the glass face *ABV* of the pyramid, and leads all the way to *M*, the mid-point of *OC*.

(ii) To comply with safety requirements, the rope used in the first stage of the contest could either be horizontal or inclined at angle to the horizontal plane not more than $30°$. Find the range of possible values of *a*. [4]

For the rest of the question, $a = 3$.

(iii) Find the coordinates of the point *X* on the face *ABV*, at which we need to drill a hole to allow the rope to penetrate through. [4]

Upon reaching the point *X*, the contestants venture into the pyramid via a glass door and continue to walk on another rope R_2 that is tied from point *X* to a finishing point on the face *OCV*. Understanding that there is a need to minimise exhaustion during this final stage of the contest, the organiser intends to let contestants take the shortest path possible.

(iv) Find the desired length of rope R_2 to be used for this final stage of the contest. [3]