# RAFFLES INSTITUTION
# 2019 YEAR 6 PRELIMINARY EXAMINATION

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|---|---|
| CANDIDATE NAME | |
| CLASS | 19 |

**MATHEMATICS** **9758/02**

PAPER 2

**3 hours**

Candidates answer on the Question Paper
Additional Materials: List of Formulae (MF26)

**READ THESE INSTRUCTIONS FIRST**

Write your name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

| FOR EXAMINER'S USE | | | | | | | |
|---|---|---|---|---|---|---|---|
| SECTION A: PURE MATHEMATICS | | | | | | | |
| Q1 | Q2 | Q3 | Q4 | Total | | | TOTAL |
| | | | | | | | |
| SECTION B: PROBABILITY AND STATISTICS | | | | | | | |
| Q5 | Q6 | Q7 | Q8 | Q9 | Q10 | Total | |
| | | | | | | | 100 |

This document consists of **7** printed pages.

**RAFFLES INSTITUTION**
Mathematics Department

## Section A: Pure Mathematics [40 Marks]

**1** The function f is defined as follows.

$$\mathrm{f} : x \mapsto \frac{1}{x^2}, \text{ for } x \in \mathbb{R},\ x \neq 0.$$

**(i)** Sketch the graph of $y = \mathrm{f}(x)$. [1]

**(ii)** If the domain of f is further restricted to $x > k$, state with a reason the least value of $k$ for which the function $\mathrm{f}^{-1}$ exists. [2]

In the rest of this question, the domain of f is $x \in \mathbb{R},\ x \neq 0$, as originally defined.

A function h is said to be an odd function if $\mathrm{h}(-x) = -\mathrm{h}(x)$ for all $x$ in the domain of h. The function g is defined as follows.

$$\mathrm{g} : x \mapsto \frac{2}{3^x - 1} + m, \text{ for } x \in \mathbb{R},\ x \neq 0.$$

**(iii)** Given that g is an odd function, find the value of $m$.

**(iv)** Using the value of $m$ found in part **(iii)**, find the range of fg.

**2** **(a)** The curve $y = \mathrm{f}(x)$ passes through the point (0, 81) and has gradient given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{1}{3}y - 15x\right)^{\frac{1}{3}}.$$

Find the first three non-zero terms in the Maclaurin series for $y$. [4

**(b)** Let $\mathrm{g}(x) = \dfrac{4 - 3x + x^2}{(1+x)(1-x)^2}$.

**(i)** Express $\mathrm{g}(x)$ in the form $\dfrac{A}{1+x} + \dfrac{1}{1-x} + \dfrac{B}{(1-x)^2}$, where $A$ and $B$ are constants to be determined.

The expansion of $\mathrm{g}(x)$, in ascending powers of $x$, is

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots + c_r x^r + \ldots.$$

**(ii)** Find the values of $c_0$, $c_1$, and $c_2$ and show that $c_3 = 3$.

**(iii)** Express $c_r$ in terms of $r$.

3 Referred to an origin $O$, the position vectors of three non-collinear points $A$, $B$ and $C$ are **a**, **b** and **c** respectively. The coordinates of $A$, $B$ and $C$ are $(-2,4,2)$, $(1,3,1)$ and $(0,1,2)$ respectively.

**(i)** Find $(\mathbf{a}-\mathbf{b})\times(\mathbf{c}-\mathbf{b})$. [1]

**(ii)** Hence

**(a)** find the exact area of triangle $ABC$,

**(b)** show that the cartesian equation of the plane $ABC$ is $3x+2y+7z=16$.

**(iii)** The line $l_1$ has equation $\mathbf{r}=\begin{pmatrix}1\\1\\6\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\-1\end{pmatrix}$ for $\lambda\in\mathbb{R}$.

**(a)** Show that $l_1$ is parallel to the plane $ABC$ but does not lie on the plane $ABC$.

**(b)** Find the distance between $l_1$ and plane $ABC$.

**(iv)** The line $l_2$ passes through $B$ and is perpendicular to the $xy$-plane. Find the acute angle between $l_2$ and its reflection in the plane $ABC$, showing your working clearly.

4 **(a)** The non-zero numbers $a$, $b$ and $c$ are the first, third and fifth terms of an arithmetic series respectively.

**(i)** Write down an expression for $b$ in terms of $a$ and $c$. [1

**(ii)** Write down an expression for the common difference, $d$, of this arithmetic series in terms of $a$ and $b$. [1

**(iii)** Hence show that the sum of the first ten terms can be expressed as

$$\frac{5}{4}(9c-a).$$ [2]

**(iv)** If $a$, $b$ and $c$ are also the fourth, third and first terms, respectively of a geometric series, find the common ratio of this series in terms of $a$ and $b$ and hence show that

$$(2b-a)a^2=b^3.$$

**(b)** The $n$th term of a geometric series is $\left(2\sin^2\alpha\right)^{n-1}$.

**(i)** Find all the values of $\alpha$, where $0\le\alpha\le 2\pi$, such that the series is convergent.

**(ii)** For the values of $\alpha$ found in part **(i)**, find the sum to infinity, simplifying your ans

**Section B: Probability and Statistics [60 Marks]**

**5** **(a)** The probability that a hospital patient has a particular disease is $p$. A test for the disease has a probability of 0.99 of giving a positive result when the patient has the disease and a probability of 0.95 of giving a negative result when the patient does not have the disease. A patient is given the test.

For a general value of $p$, the probability that a randomly chosen patient has the disease given that the result of the test is positive is denoted by $\mathrm{f}(p)$.

Find an expression for $\mathrm{f}(p)$ and show that f is an increasing function for $0 < p < 1$. Explain what this statement means in the context of this question. [5

**(b)** For events $A$ and $B$, it is given that $\mathrm{P}(A) = 0.7$ and $\mathrm{P}(B) = k$.

**(i)** Given that $A$ and $B$ are independent events, find $\mathrm{P}(A \cap B')$ in terms of $k$.

**(ii)** Given instead that $A$ and $B$ are mutually exclusive events, state the range of values of $k$.

Find $\mathrm{P}(B \mid A')$ in terms of $k$.

**6** Five objects $a, b, c, d$ and $e$ are to be placed in five containers $A, B, C, D$ and $E$, with one in each container. An object is said to be correctly placed if it is placed in the container of the same letter (e.g. $a$ in $A$) but incorrectly placed if it is placed in any of the other four containers. Find

**(i)** the number of ways the objects can be placed in the containers so that $a$ is correctly placed and $b$ is incorrectly placed, [2]

**(ii)** the number of ways the objects can be placed in the containers so that both $a$ and $b$ are incorrectly placed, [3]

**(iii)** the number of ways the objects can be placed in the containers so that there are at least 2 correct placings. [3]

**7** Based on past records, at government polyclinics, on average each medical consultation lasts 15 minutes with a standard deviation of 10 minutes.

**(i)** Give a reason why a normal distribution, with this mean and standard deviation, would not give a good approximation to the distribution of consultation duration. [2

From recent patient and doctor feedback, a polyclinic administrator claims that the average consultation duration at government polyclinic is taking longer. State suitable null and alternative hypotheses to test this claim.

**(ii)** Given that the null hypothesis will be rejected if the sample mean from a random sample of size 30 is at least 18, find the smallest level of significance of the test. State clearly any assumptions required to determine this value.

The administrator suspects the average consultation duration at private clinics is actually less than 15 minutes. A survey is carried out by recording the consultation duration, $y$, in minutes, from 80 patients as they enter and leave a consultation room at a private clinic. The results are summarized by

$$\sum(y-15)=-50\,,\ \sum(y-15)^2=555\,.$$

**(iii)** Calculate unbiased estimates of the population mean and variance of the consultation duration at private clinics. Determine whether there is sufficient evidence at the 5% level of significance to support the administrator's claim.

**8** A biased cubical die has its faces marked with the numbers 1, 3, 5, 7, 11 and 13. The random variable $X$ is defined as the score obtained when the die is thrown, with probabilities given by

$$\mathrm{P}(X=r)=kr\,,\ r=1,3,5,7,11,13,$$

where $k$ is a constant.

**(i)** Show that $\mathrm{P}(X=3)=\dfrac{3}{40}$. [3]

**(ii)** Find the exact value of Var($X$).

The die is thrown 15 times and the random variable $R$ denotes the number of times that a score less than 10 is observed.

**(iii)** Find $\mathrm{P}(R \geq 5)$.

**(iv)** Find the probability that the last throw is the $8^{\text{th}}$ time that a score less than 10 is observed.

**9** Rroar Tyre Company develops Brand R tyres. The working lifespan in kilometres of a Brand R tyre is a random variable with the distribution $N(64000, 8000^2)$.

**(i)** Find the probability that a randomly selected Brand R tyre has a working lifespan of at least 70000 km. [1

**(ii)** Rroar Tyre Company wishes to advertise that 98% of their Brand R tyres have working lifespans of more than $t$ kilometres. Determine the value of $t$, correct to the nearest kilometre. [2

Ssoar Tyre Company, a rival company, develops Brand S tyres. The working lifespan in kilometres of a Brand S tyre is a random variable with the distribution $N(68000, \alpha^2)$.

**(iii)** A man selects 50 Brand S tyres at random. Given that $\alpha = 7500$, find the probability that the average of their working lifespans exceeds 70000 km.

**(iv)** Using $\alpha = 8000$, find the probability that the sum of the working lifespans of 3 randomly chosen Brand R tyres is less than 3 times the working lifespan of a randomly chosen Brand S tyre.

**(v)** Find the range of $\alpha$, correct to the nearest kilometre, if there is a higher percentage of Brand S tyres than Brand R tyres lasting more than 50000 km.

**(vi)** State clearly an assumption needed for your calculations in parts **(iii)**, **(iv)** and **(v)**.

**10** **(a)** With the aid of suitable diagrams, describe the differences between the least square linear regression line of $y$ on $x$ and that of $x$ on $y$. [2]

**(b)** The government of the Dragon Island Country is doing a study on its population growth in order to implement suitable policies to support the aging population of the country. The population sizes, $y$ millions in $x$ years after Year 2000, are as follows.

| $x$ | 9 | 10 | 11 | 12 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|
| Population size, $y$ | 5.05 | 5.21 | 5.32 | 5.41 | 5.51 | 5.56 | 5.60 | 5.65 | 5.67 |

**(i)** Draw a scatter diagram of these values, labelling the axes clearly. Use your diagram to explain whether the relationship between $x$ and $y$ is likely to be well modelled by an equation of the form $y = ax + b$, where $a$ and $b$ are constants.

**(ii)** Find, correct to 6 decimal places, the value of the product moment correlation coefficient between

**(a)** $\ln x$ and $y$,

**(b)** $x^2$ and $y$.

**(iii)** Use your answers to part **(ii)** to explain which of $y = a + b\ln x$ or $y = a + bx^2$ is the better model.

**(iv)** It is required to estimate the year in which the population size first exceed 6.5 millions. Use the model that you identified in part **(iii)** to find the equation of a suitable regression line, and use your equation to find the required estimate.

Comment on the reliability of this estimate.

**(v)** As the population size in Year 2013 is not available, a government statistician uses the regression line in part **(iv)** to estimate the population size in 2013. Find this estimate.

**(vi)** It was later found that in Year 2013, the population size was in fact 5.31 millions. Comment on this figure with reference to the estimate found in part **(v)**, providing a possible reason in context.