ANDERSON SERANGOON JUNIOR COLLEGE

MATHEMATICS **9758**

H2 Math Prelim Paper 1 (100 marks) **12 Sept 2022**

3 hours

Additional Material(s): List of Formulae (MF26)

CANDIDATE NAME

CLASS

READ THESE INSTRUCTIONS FIRST

Write your name and class in the boxes above.
Please write clearly and use capital letters.
Write in dark blue or black pen. HB pencil may be used for graphs and diagrams only.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions and write your answers in this booklet.
Do not tear out any part of this booklet.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

All work must be handed in at the end of the examination. If you have used any additional paper, please insert them inside this booklet.
The number of marks is given in brackets [ ] at the end of each question or part question.

Question number	Marks
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
Total	

This document consists of 22 printed pages and 2 blank pages.

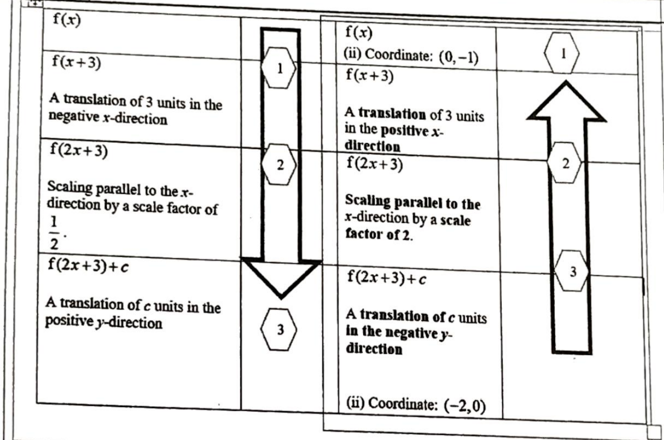
[Turn Over

1	(i)	For positive real constant c, state a sequence of three transformations in terms of c, that will transform the graph with equation of the form $y=\mathrm{f}(2x+3)+c$ onto the graph with equation $y=\mathrm{f}(x)$.	[3]
	(ii)	The point with coordinates $(-2,0)$ that lies on the curve with equation of the form $y=\mathrm{f}(2x+3)+c$ is mapped onto the point with coordinates $(0,-1)$ that is on the curve with equation $y=\mathrm{f}(x)$. State the value of c.	[1]

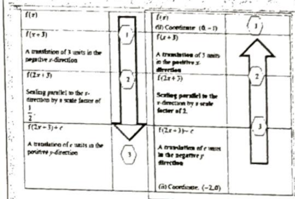
Solution

(i) A translation of c units in the negative y direction.
Scaling parallel to the x-axis by a scale factor of 2.
A translation of 3 units in the positive x-direction.

(ii) $c=1$



Commented [SH1]: Question reading / Presentation / Conceptual understanding



- Standard phrasing must be followed. Students should make it a point to memorise the phrasing for Translation / Reflection/ Scaling. Marks should not be lost in these kind of questions.

Commented [SH2]: Conceptual Understanding

Coordinate (-2,0) undergoes the ***only 1 transformation which affects the y-axis*** of which is the ***Translation of c units in the negative y-direction*** to Coordinate (0, -1)
Therefore: c =1

2		The complex numbers z_1, z_2 and z_3 are given by $z_1=\left(1-\sqrt{3}\mathrm{i}\right)^2$, $z_2=\left[\sqrt{2}\left(\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}\right)\right]^6$ and $z_3=-1+\sqrt{3}\mathrm{i}$.	
	(i)	Using an algebraic method, find $\frac{z_2}{z_1}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and θ is an exact real constant such that $-\pi<\theta\le\pi$.	[3]
	(ii)	Hence find $\frac{z_2}{z_1}+z_3$ in the form $p\mathrm{e}^{\mathrm{i}\alpha}$, where both r and θ are exact real constants such that $r>0$ and $-\pi<\theta\le\pi$.	[3]

Solution

(i) $\dfrac{z_2}{z_1} = \dfrac{\left[\sqrt{2}e^{\left(\frac{\pi}{4}\right)i}\right]^6}{\left[2e^{\left(\frac{\pi}{3}\right)i}\right]^2}$

$\dfrac{z_2}{z_1} = 2e^{\left(\frac{13\pi}{6}\right)i}$

$\therefore \dfrac{z_2}{z_1} = 2e^{\left(\frac{\pi}{6}\right)i}$

(ii) $\dfrac{z_2}{z_1} + z_3 = 2e^{\left(\frac{\pi}{6}\right)i} + 2e^{\left(\frac{2\pi}{3}\right)i}$

$= 2e^{\left(\frac{\frac{\pi}{6}+\frac{2\pi}{3}}{2}\right)i}\left[e^{\left(\frac{\frac{\pi}{6}-\frac{2\pi}{3}}{2}\right)i} + e^{\left(\frac{\frac{\pi}{6}-\frac{2\pi}{3}}{2}\right)i}\right]$

$= 2e^{\left(\frac{5\pi}{12}\right)i}\left[2\cos\left(-\frac{\pi}{4}\right)\right]$

$= 2\sqrt{2}e^{\left(\frac{5\pi}{12}\right)i}$

Commented [LT3]: Misconception
Did not obtain the correct exponential form for the complex numbers given.

Recommendation
1. Locate the point in the Argand Diagram before evaluating its argument.
2. Whenever possible, use exponential form to perform any simplifications. Using polar form for any simplification is strongly discourage.

Commented [LT4]: Question Reading
It is important to have the habit of leaving the final argument value of the complex number to be within the principal range.

Commented [LT5]: Misconception
Did not obtain the correct exponential form for z_3.

Commented [LT6]: Recommendation
Majority could not remember the properties learnt in the lecture. It is important to remember them.

Commented [LT7]: Presentation of Answer
Final answer has to be in the simplest form whenever possible.

3 It is given that the curve C has equation $y = \dfrac{x^2 - x + 7}{x - 2}$, $x \in \mathbb{R}$, $x \neq 2$.

(i) Without using a calculator, find the set of values that y cannot take. [3]

(ii) Sketch C, stating clearly the equations of any asymptotes, the coordinates of the stationary points and the point(s) where the curve crosses the axes. [3]

Solution

(i) $y = \dfrac{x^2 - x + 7}{x - 2}$

Method 1: $x^2 - x + 7 = y(x - 2)$

$x^2 - (1 + y)x + 7 + 2y = 0$

For the equation to not have real solutions, discriminant < 0

$[-(1+y)]^2 - 4(7 + 2y) < 0$

$y^2 - 6y - 27 < 0$

$(y - 9)(y + 3) < 0$

$-3 < y < 9$

$\therefore$ The set of values that C cannot take is $\{y \in \mathbb{R} : -3 < y < 9\}$.

Method 2:

$y = x+1+\frac{9}{x-2}$

$\frac{dy}{dx} = 1-\frac{9}{(x-2)^2} = 0$ (for stationary points)

$x-2 = 3 \text{ or } -3$

$x = 5 \text{ or } -1$

$y = 9 \text{ or } -3$

$\frac{d^2y}{dx^2} = \frac{18}{(x-2)^3}$

When x = 5, $\frac{d^2y}{dx^2} = \frac{2}{3} > 0 \Rightarrow (5,9)$ is a minimum point

When x = −1, $\frac{d^2y}{dx^2} = -\frac{2}{3} < 0 \Rightarrow (-1,-3)$ is a maximum point

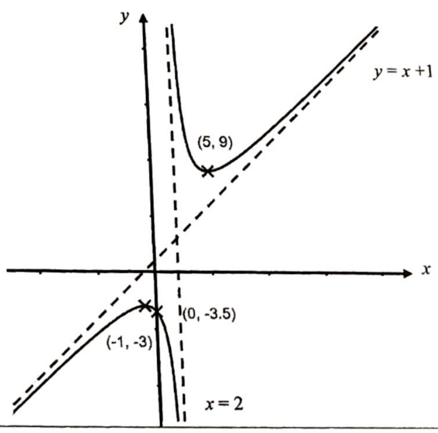
The curve is undefined at $x = 2$. For $x > 2$, $\frac{d^2y}{dx^2} > 0 \Rightarrow$ curve is concave upwards

For $x < 2$, $\frac{d^2y}{dx^2} < 0 \Rightarrow$ curve is concave downwards

Hence $y \geq 9$ or $y \leq -3$

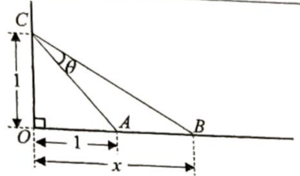
$\therefore$ The set of values that C cannot take is $\{y \in \mathbb{R} : -3 < y < 9\}$

(ii) $y = \frac{x^2-x+7}{x-2} = x+1+\frac{9}{x-2}$



Commented [KSM8]: Strategy
When the differentiation method is used and ar algebraic method is required, you must explain the shape of the curve in all regions of x. Just showing the existence of stationary points is insufficient.

4 **(i)** Show that the first two non-zero terms of the Maclaurin series for $\tan\theta$ is given by $\theta+\frac{1}{3}\theta^3$. You may use the standard results given in the List of Formulae (MF26). [2]



In the right-angle triangle OBC shown above, point A lies on OB such that $OA=1$, $OB=x$, where $x>1$ and $OC=1$. It is given that angle COB is $\frac{\pi}{2}$ radians and that angle ACB is θ radians (see diagram).

(ii) Show that $AB=\dfrac{2\tan\theta}{1-\tan\theta}$. [2]

(iii) Given that θ is a sufficently small angle, show that

$$AB\approx a\theta+b\theta^2+c\theta^3$$

for exact real constants a, b and c to be determined. [3]

Solution

(i)

$$\tan\theta=\frac{\sin\theta}{\cos\theta}$$

$$=\sin\theta(\cos\theta)^{-1}$$

$$\approx\left(\theta-\frac{\theta^3}{3!}\right)\left(1-\frac{\theta^2}{2!}\right)^{-1}$$

$$\approx\left(\theta-\frac{\theta^3}{3!}\right)\left(1+\frac{\theta^2}{2!}\right)$$

$$\approx\theta+\frac{\theta^3}{2!}-\frac{\theta^3}{3!}$$

$$=\theta+\frac{1}{3}\theta^3$$

(ii) $\tan\left(\frac{\pi}{4}+\theta\right)=\dfrac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}\tan\theta}$

$$x=\frac{1+\tan\theta}{1-\tan\theta}$$

$$AB=\frac{1+\tan\theta}{1-\tan\theta}-1$$

Commented [KSX9]: Presentation
Use approximate sign if you are writing down first few terms of the series only.

Misconception

$$\frac{\left(\theta-\frac{\theta^3}{3!}\right)}{\left(1-\frac{\theta^2}{2!}\right)}=\theta-\frac{\theta}{3}$$

Commented [KSX10]: Due to lack of practice, many students did not know they have to do this step.

Commented [KSX11]: Method
Students can also approach this question using sine rule.

[Turn Over

$AB = \dfrac{2\tan\theta}{1-\tan\theta}$

(iii) $AB = 2\tan\theta(1-\tan\theta)^{-1}$

$\approx 2\left(\theta+\dfrac{\theta^3}{3}\right)\left(1-\left(\theta+\dfrac{\theta^3}{3}\right)\right)^{-1}$

$\approx 2\left(\theta+\dfrac{\theta^3}{3}\right)\left(1+\left(\theta+\dfrac{\theta^3}{3}\right)+\left(\theta+\dfrac{\theta^3}{3}\right)^2\right)$

$\approx \left(2\theta+\dfrac{2\theta^3}{3}\right)\left(1+\theta+\theta^2\right)$

$\approx 2\theta+2\theta^2+\dfrac{8\theta^3}{3}$

$a=2,\ b=2,\ c=\dfrac{8}{3}$

Commented [KSX12]: As the question asked to show first three term of the series, students should approximate $\tan\theta$ to $\theta+\dfrac{\theta^3}{3}$.

Commented [KSX13]: Presentation
Students who managed to get to the correct answer did not state the values of a, b and c.

5 **(i)** By considering $u_n - u_{n+1}$, where $u_n = \dfrac{1}{n(n+1)(n+2)}$,

find $\displaystyle\sum_{n=1}^{N}\frac{1}{n(n+1)(n+2)(n+3)}$ in terms of N. [3]

(ii) Hence or otherwise, find $\displaystyle\sum_{n=5}^{N+3}\frac{1}{n(n-1)(n-2)(n-3)}$. [3]

(iii) Deduce that

$$\frac{1}{6^2}+\frac{1}{12^2}+\frac{1}{20^2}+\frac{1}{30^2}+\frac{1}{42^2}+\ldots$$

is less than $\dfrac{1}{18}$. Show your workings clearly. [3]

Solution

(i)

Commented [ABK14]: Approach
When a question states "**by considering…**", we must use the approach by looking at the suggested expression $u_n - u_{n+1}$ and work from here to solve this summation problem.

Commented [ABK15]: Approach/ Strate
With a statement like "**hence**", typically this approach is the best way to solve the probler The suggested "**otherwise**" approach can als used but often, it may not be the most efficie method to adopt under examination time constraint. So this has got to do with exam strategy. For the "**hence**" approach, use of t previous result is necessary.

Commented [ABK16]: Approach
With a statement like "**deduce**", we must st use the earlier results to prove this part of t question. We must relate clearly how this p the question use the previous results to arr the final solution.

$$u_n - u_{n+1} = \frac{1}{n(n+1)(n+2)} - \frac{1}{(n+1)(n+2)(n+3)}$$

$$= \frac{(n+3)-n}{n(n+1)(n+2)(n+3)}$$

$$\Rightarrow u_n - u_{n+1} = \frac{3}{n(n+1)(n+2)(n+3)}$$

$$\sum_{n=1}^{N} \frac{1}{n(n+1)(n+2)(n+3)}$$

$$= \frac{1}{3}\sum_{n=1}^{N}[u_n - u_{n+1}]$$

$$= \frac{1}{3}[\ u_1 - u_2 + u_2 - u_3 + u_3 - u_4 + \ldots + u_{N-1} - u_N + u_N - u_{N+1}]$$

Alternative way (discouraged)

$$= \frac{1}{3}\Big[\frac{1}{6} - \frac{1}{24} + \frac{1}{24} - \frac{1}{60} + \frac{1}{60} - \frac{1}{120} + \ldots + \frac{1}{(N-1)(N)(N+1)} - \frac{1}{(N)(N+1)(N+2)} + \frac{1}{(N)(N+1)(N+2)} - \frac{1}{(N+1)(N+2)(N+3)}\Big]$$

$$= \frac{1}{3}[u_1 - u_{N+1}]$$

$$= \frac{1}{3}\left[\frac{1}{(1)(2)(3)} - \frac{1}{(N+1)(N+2)(N+3)}\right]$$

$$= \frac{1}{18} - \frac{1}{3(N+1)(N+2)(N+3)}$$

(ii) By replacing n with $(n+3)$,

$$\sum_{n=5}^{N+3} \frac{1}{n(n-1)(n-2)(n-3)} = \sum_{n+3=5}^{n+3=N+3} \frac{1}{(n+3)(n+3-1)(n+3-2)(n+3-3)}$$

$$= \sum_{n=2}^{N} \frac{1}{(n)(n+1)(n+2)(n+3)}$$

$$= \sum_{n=1}^{N} \frac{1}{(n)(n+1)(n+2)(n+3)} - \sum_{n=1}^{1} \frac{1}{(n)(n+1)(n+2)(n+3)}$$

$$= \left[\frac{1}{18} - \frac{1}{3(N+1)(N+2)(N+3)}\right] - \left[\frac{1}{(1)(2)(3)(4)}\right]$$

$$= \frac{1}{72} - \frac{1}{3(N+1)(N+2)(N+3)}$$

(iii) For positive integers n

Commented [ABK17]: Approach
In evaluating this summation using the Method of Difference, **avoid substituting values** for each term in this case. The approach is clear that we are using $u_n - u_{n+1}$. If we insist in evaluating each value, do note that the method is not wrong but it is **NOT EFFICIENT** under examination time constraint.

Commented [ABK18]: Technique
We have solved this in part (i) $\sum_{n=1}^{N} \frac{1}{n(n+1)(n+2)(n+3)} = \frac{1}{18} - \frac{1}{3(N+1)(N+2)(N+3)}$. For part (ii), taking the "**hence**" approach, we are to use the above result.

For such a question the technique is to use "**replacement of n**". We should always start from the question that we are targeting. In this case we are "**replacing n with (n+3)**". Why "n+3"? The simple reason is that we want to make the current expression looks the same as that found in part (i) so that we can use its result.

Commented [ABK19]: Continuation from above:
At this juncture, the expression within the summation is now the same as that of part (i). Now, we need to split the limits to apply the result in part (i) correctly. Remember the "**cutting the cake**" method – basically counting the terms.

$n^2+3n<n^2+3n+2$

$n(n+3)<(n+1)(n+2)$

$n(n+1)(n+2)(n+3)<(n+1)^2(n+2)^2$

$\therefore \dfrac{1}{n(n+1)(n+2)(n+3)}>\dfrac{1}{(n+1)^2(n+2)^2} \quad \forall n>0$

So $\displaystyle\sum_{n=1}^{N}\frac{1}{(n+1)^2(n+2)^2}<\sum_{n=1}^{N}\frac{1}{n(n+1)(n+2)(n+3)}$

As $\displaystyle\frac{1}{6^2}+\frac{1}{12^2}+\frac{1}{20^2}+\ldots=\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{(n+1)^2(n+2)^2}$

$$<\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n(n+1)(n+2)(n+3)}$$

$$=\lim_{N\to\infty}\left[\frac{1}{18}-\frac{1}{3(N+1)(N+2)(N+3)}\right]$$

$$=\frac{1}{18}$$

Thus $\dfrac{1}{6^2}+\dfrac{1}{12^2}+\dfrac{1}{20^2}+\ldots<\dfrac{1}{18}$ (deduced)

6	(a) Find $\displaystyle\int\frac{\sin^{-1}(2x-1)}{\sqrt{1-x}}\,dx$ for $0<x<1$.	[3]
	(b) (i) Sketch the graphs of $y=\lvert x^2-7\rvert$ and $y=x+5$ on the same diagram. Indicate clearly the x-intercepts and the values of x where the two curves intersect. Hence solve the inequality $\lvert x^2-7\rvert\ge x+5$.	[4]
	(ii) Hence, for $a>5$, find $\displaystyle\int_3^a\left\lvert\,\lvert x^2-7\rvert-x-5\right\rvert\,dx$ in terms of a. Leave your answer in exact form.	[3]

Solution

(a)

Commented [ABK20]: Method
This inequality must be **established** before we can proceed with the next step. **To even think** about this inequality we must first identify the expression for this sum to infinity:

$\dfrac{1}{6^2}+\dfrac{1}{12^2}+\dfrac{1}{20^2}+\ldots=\displaystyle\sum_{n=1}^{\infty}\frac{1}{(n+1)^2(n+2)^2}$.

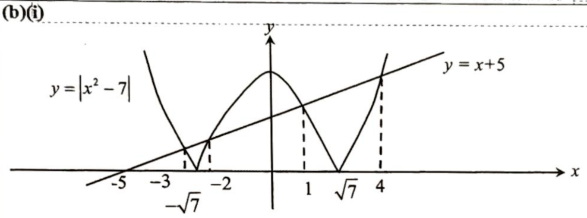
Looking at the RHS, that means we need to establish a link between $(n+1)^2(n+2)^2$ and $n(n+1)(n+2)(n+3)$ which is found in our original expression. This is the start of our **thinking process**.

Also, we know that we need to show $\dfrac{1}{6^2}+\dfrac{1}{12^2}+\dfrac{1}{20^2}+\ldots=\displaystyle\sum_{n=1}^{\infty}\frac{1}{(n+1)^2(n+2)^2}<\frac{1}{18}$.
The only $\frac{1}{18}$ that we can find is from part (i).
This would give us more clue to establish a link (inequality) between $(n+1)^2(n+2)^2$ and $n(n+1)(n+2)(n+3)$.

Commented [KW(W21]: Question Reading
$\displaystyle\int\frac{\sin^{-1}(2x-1)}{\sqrt{1-x}}\,dx$ (indefinite integral) for $0<x<1$ is not equivalent to $\displaystyle\int_0^1\frac{\sin^{-1}(2x-1)}{\sqrt{1-x}}\,dx$ (definite integral).

$$\int \frac{\sin^{-1}(2x-1)}{\sqrt{1-x}}\,dx = -2\sqrt{1-x}\cdot\sin^{-1}(2x-1) + \int 2\sqrt{1-x}\,\frac{2}{\sqrt{1^2-(2x-1)^2}}\,dx$$
$$= -2\sqrt{1-x}\cdot\sin^{-1}(2x-1) + 4\int \frac{\sqrt{1-x}}{\sqrt{1-x}\sqrt{4x}}\,dx$$
$$= -2\sqrt{1-x}\cdot\sin^{-1}(2x-1) + 2\int (x)^{-\frac{1}{2}}\,dx$$
$$= -2\sqrt{1-x}\sin^{-1}(2x-1) + 4\sqrt{x} + C$$

(b)(i)



From the sketch, $x \le -3$ or $-2 \le x \le 1$ or $x \ge 4$

(ii) $\int_3^a \left|\,|x^2-7|-x-5\,\right| dx = \int_3^4 x+5-|x^2-7|\,dx + \int_4^a |x^2-7|-x-5\,dx$

$$= \frac{19}{6} + \int_4^a x^2-7\,dx - \int_4^a x+5\,dx$$

$$= \frac{19}{6} + \left[\frac{x^3}{3}-7x\right]_4^a - \left[\frac{x^2}{2}+5x\right]_4^a$$

$$= \frac{19}{6} + \left(\frac{a^3}{3}-7a\right) + \frac{20}{3} - \left(\frac{a^2}{2}+5a\right) + 28$$

$$= \frac{a^3}{3} - \frac{a^2}{2} - 12a + \frac{227}{6}$$

7 A curve C has parametric equations

$$x = \sin^3 t\,, \quad y = \cos^2 t,\ -\frac{\pi}{2} < t < 0.$$

The tangent at the point $P(\sin^3 p, \cos^2 p)$, $-\frac{\pi}{2} < p < 0$, meets the x-axis and y-axis at Q and R respectively.

(i) By finding the equation of the tangent at the point P, show that the area of the triangle OQR is $-\frac{1}{12}\sin p\,(2+\cos^2 p)^2$. [6]

(ii) Find a cartesian equation of the locus of the mid-point of QR as p varies. You need not indicate its domain. [5]

Solution

Commented [KW(W22]: Techniques
There were many mistakes observed for this part:
- Some students did not know that they need to use integration by parts.
- Others were not able to choose u and dv/dx correctly.
- Some others were not able to differentiate $\sin^{-1}(2x-1)$ and/or integrate $\frac{1}{\sqrt{1-x}}$ and/or $\frac{1}{\sqrt{x}}$ correctly.

Commented [KW(W23]: Presentation
The graph of $y=|x^2-7|$ is symmetrical about the y-axis. Some students could not draw the graph correctly for $x<-\sqrt{7}$ and $x>\sqrt{7}$.

Commented [KW(W24]: Concept/Presentation
Some students do not know when to use 'or' a[nd] 'and'. Others use ',' instead of 'or'.

Commented [KW(W25]: Method
Students should use part (ii) to determine how [to] split the interval from 3 to a and the sign of $\left|\,|x^2-7|-x-5\right|$ in the respective intervals.

[Turn Over

(i)

$x=\sin^3 t$ $\quad$ $y=\cos^2 t$

$\frac{dx}{dt}=3\sin^2 t\cos t$ $\quad$ $\frac{dy}{dt}=-2\sin t\cos t$

$\frac{dy}{dx}=\frac{-2\sin t\cos t}{3\sin^2 t\cos t}=-\frac{2}{3\sin t}$

At the point P, $x=\sin^3 p$

$y=\cos^2 p$

$\frac{dy}{dx}=-\frac{2}{3\sin p}$

Equation of the tangent at the point P:

$$y-\cos^2 p=-\frac{2}{3\sin p}(x-\sin^3 p)$$

When $y=0$, $-\cos^2 p=-\frac{2}{3\sin p}(x-\sin^3 p)$

$x=\sin^3 p+\frac{3}{2}\sin p\cos^2 p$

$x=\frac{1}{2}\sin p(2\sin^2 p+3\cos^2 p)$

$x=\frac{1}{2}\sin p(2+\cos^2 p)$

$Q\left(\frac{1}{2}\sin p(2+\cos^2 p),0\right)$

When $x=0$, $y-\cos^2 p=-\frac{2}{3\sin p}(0-\sin^3 p)$

$y=\frac{2}{3}\sin^2 p+\cos^2 p$

$y=\frac{1}{3}(2\sin^2 p+3\cos^2 p)=\frac{1}{3}(2+\cos^2 p)$

$R\left(0,\frac{1}{3}(2+\cos^2 p)\right)$

Area of the triange OQR

$=\frac{1}{2}\times OQ\times OR$

$=\frac{1}{2}\left|\sin p(2+\cos^2 p)\right|\times\frac{1}{3}\left|2+\cos^2 p\right|$

$=\frac{1}{12}\left|\sin p\right|\left(2+\cos^2 p\right)^2$

$=-\frac{1}{12}\sin p\left(2+\cos^2 p\right)^2$ $\left(\sin p<0 \because -\frac{\pi}{2}<p<0\right)$

(ii)

Commented [CKJ26]: Observation
Many students did a conversion using double angle formula before differentiation. Some students continue to modify the expression using factor formula which was not necessary. Students should know how to differentiate the given expression directly.

Commented [CKJ27]: Common Mistake
Some students did not know that they had to find the gradient at the point P. Some students gave the equation of tangent at P as $y-\cos^2 p=-\frac{2}{3\sin t}(x-\sin^3 p)$ which was incorrect.

Commented [CKJ28]: Common Mistake
Many students did not realise the x coordinates of Q is negative or sin p < 0. To remove the modulus sign, students need to introduce a minus sign in front.

	Mid point of $QR = \left(\dfrac{\frac{1}{2}\sin p(2+\cos^2 p)+0}{2}, \dfrac{0+\frac{1}{3}(2+\cos^2 p)}{2}\right)$ $=\left(\frac{1}{4}\sin p(2+\cos^2 p), \frac{1}{6}(2+\cos^2 p)\right)$
	$x=\frac{1}{4}\sin p(2+\cos^2 p)$ ------------ (1) $y=\frac{1}{6}(2+\cos^2 p)$ ------------ (2)
	$\frac{(1)}{(2)}$ gives
	$\frac{x}{y}=\dfrac{\frac{1}{4}\sin p(2+\cos^2 p)}{\frac{1}{6}(2+\cos^2 p)}$ $\frac{x}{y}=\frac{3}{2}\sin p$ $\sin p=\frac{2x}{3y}$
	$y=\frac{1}{6}(2+\cos^2 p)$ $y=\frac{1}{6}\left(2+(1-\sin^2 p)\right)$ $y=\frac{1}{6}\left(3-\frac{4x^2}{9y^2}\right)$
	$y=\frac{1}{54y^2}\left(27y^2-4x^2\right)$
	$54y^3=27y^2-4x^2$ Cartesian equation of the locus of the mid-point of QR is $54y^3=27y^2-4x^2$

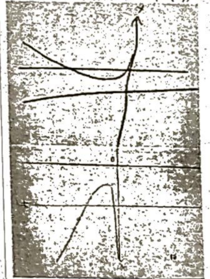
Commented [CKJ29]: Observation
This question was poorly attempted. Many students did not attempt the question.

Approach
The idea is to find the mid point of QR. Expr
and y in terms of p. Then think of a way to
of the parameter p.

[Turn Over

8	**(a)** Functions f and g are defined by $\mathrm{f}: x \mapsto x^2, \quad x<0,$ $\mathrm{g}: x \mapsto \frac{1}{x}, \quad x>0.$	
	(i) Explain why the composite function gf exists.	[1]
	(ii) Find the exact value of $\mathrm{f}^{-1}\mathrm{g}^{-1}(3)$. Show your workings clearly.	[3]
	(b) For real values a, the function h is defined by $\mathrm{h}: x \mapsto ax - \frac{1}{x}, \quad x<0.$	
	(i) If a is negative, explain clearly with a well-labelled diagram, why h^{-1} does not exist.	[4]
	(ii) If $a=1$, find h^{-1} in similar form.	[3]
	Solution	
	(ai) $R_f = (0,\infty)$ and $D_g = (0,\infty)$	
	Since $R_f \subseteq D_g$, the composite function gf exists.	
	(ii) Let $\mathrm{f}^{-1}\mathrm{g}^{-1}(3) = k$	
	$\mathrm{g}^{-1}(3) = \mathrm{f}(k) = k^2$ ------(1)	
	$\mathrm{g}(k^2) = 3$	
	$\frac{1}{k^2} = 3$	

Commented [LT30]: Question Reading
Many did not comprehend what it means to be a well-labelled diagram. Some bad examples are shown below. One needs to indicate the key features of the curve like turning point(s), asymptote(s), intercept(s), if any.



Commented [LT31]: Presentation of Answer
It is important to tell the marker what the individual range and domain were before making the conclusion.

Commented [LT32]: Misconception
Some did not indicate the equal sign in this statement made.

Commented [LT33]: Recommendation
$\mathrm{f}^{-1}\mathrm{g}^{-1}(3) = k$

$\Rightarrow \mathrm{f}\mathrm{f}^{-1}\mathrm{g}^{-1}(3) = \mathrm{f}(k) \quad \Rightarrow \mathrm{g}^{-1}(3) = k^2$

So one need not find the composite function to do this question.

Misconception

$\mathrm{f}^{-1}\mathrm{g}^{-1}(x) \neq (\mathrm{fg})^{-1}(x)$

In fact $\mathrm{f}^{-1}\mathrm{g}^{-1}(x) = (\mathrm{gf})^{-1}(x)$

$k = -\frac{\sqrt{3}}{3}\ (\because D_f = (-\infty, 0))$

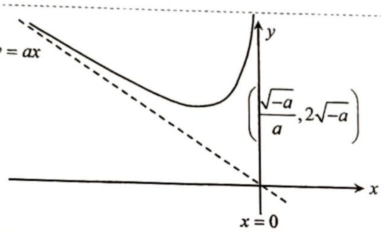
(bi)

$h(x) = ax - \frac{1}{x}$

$h'(x) = a + \frac{1}{x^2}$

For $a + \frac{1}{x^2} = 0 \Rightarrow x = \frac{-1}{\sqrt{-a}} = \frac{\sqrt{-a}}{a}\ (\because x < 0)$

$\therefore h\left(\frac{\sqrt{-a}}{a}\right) = \sqrt{-a} + \sqrt{-a} = 2\sqrt{-a}$



Since the horizontal line $y = 2\sqrt{-a} + 1$ cuts the curve twice, the function is not a 1-1 function and so h^{-1} does not exist.

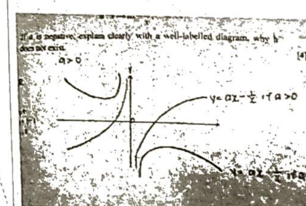
Commented [LT34]: Misconception
Many forgot that k must be in the domain o equation (1) to work.

Commented [LT35]: Misconception
A number did not identify the correct x valu the turning point.

Commented [LT36]: Presentation
A number did not simplify the answer.

Note that $\frac{-a}{\sqrt{-a}} = \frac{\left(\sqrt{-a}\right)^2}{\sqrt{-a}} = \sqrt{-a}$

Commented [LT37]: Presentation of and Question Reading
Many failed to indicate the key features o curve (turning point, asymptotes) and som not even draw the required curve within domain stated. One such bad example is shown below.



Commented [LT38]: Misconception
To disprove that it is a one-one function sufficient to suggest one particular hori line that violates the horizontal line test Many wrote "**for all lines $y=k$, k is a rea the line cuts the curve more than once** is not true for line y=0.

	(ii) Let $y = \mathrm{h}(x) = x - \frac{1}{x}$		
	$y = x - \frac{1}{x}$ $yx = x^2 - 1$		
	$x^2 - xy - 1 = 0$		
	$x = \frac{y \pm \sqrt{y^2+4}}{2}$		
	$x = \frac{y - \sqrt{y^2+4}}{2}$ $(\because x < 0)$		
	$\mathrm{h}^{-1} : x \mapsto \frac{x - \sqrt{x^2+4}}{2}$, $x \in \mathbb{R}$		
9	**(a)** An arithmetic progression has first term a and common difference d, where $a > 0$ and $d \neq 0$. The eighth, third and second term of the progression are the first three terms of an infinite geometric progression.		
	(i) Find the common ratio of the geometric progression.	[3]	
	(ii) Find the exact sum of the odd-numbered terms of the geometric progression in terms of a.	[3]	
	(b) A programmer coded a program involving a rabbit-fox chase along a straight path to model the actual hunt for a rabbit by a fox.		
	The rabbit first hop is 1.75 m. In each subsequent hop, the distance covered is 1% less than its previous hop. The fox first leaps 3 m. In each subsequent leap, the distance covered is 0.02 m less than its previous leap. Initially the rabbit is 60 m ahead of the fox and assume that the rabbit and the fox start and end each hop and leap at the same time.		
	(i) By finding the total distance travelled by the fox and the rabbit after n leaps and hops respectively, find the minimum number of hops and leaps for the fox to catch up with the rabbit.	[4]	
	(ii) Find the number of leaps the fox takes before it comes to a stop. Hence, find the minimum starting distance, in metre, between the fox and the rabbit such that the fox will never catch up with the rabbit. Leave your answer to the nearest integer.	[2]	
	Solution		
	(ai) Let b and r be the first term and common ratio of the G.P.		
	$b = a + 7d$ ---------------------- (1) $br = a + 2d$ ---------------------- (2) $br^2 = a + d$ ---------------------- (3)		
	From (1) and (2) gives $b - br = 5d$ ---------------------- (4) From (2) and (3) gives $br - br^2 = d$ ---------------------- (5)		
	(4) divides (5) gives		

Commented [LT39]: Misconception
A number did not select the correct equation that is based on the domain of h(x).

Commented [LT40]: Question Reading
A number did not express the answer in the similar form. It must be written in the form as how the question has presented.

$\frac{1-r}{r-r^2}=5$

$5r^2-6r+1=0$

$(5r-1)(r-1)=0$

$r=\frac{1}{5}$ or $r=1$ (rejected since $d\neq 0$)

(ii) From (5), $\frac{4}{25}b=d$.

And from (3), $b=-\frac{25}{3}a$

$(S_\infty)_{\text{odd}}=\frac{b}{1-r^2}$

$=\frac{-\frac{25}{3}a}{1-\frac{1}{25}}$

$=-\frac{625a}{72}$

(bi) $(S_n)_{\text{fox}}=\frac{n}{2}[2(3)+(n-1)(-0.02)]=n(3.01-0.01n)$

$(S_n)_{\text{rabbit}}=\frac{1.75[1-0.99^n]}{1-0.99}=175(1-0.99^n)$

For the fox to catch the rabbit,

$n(3.01-0.01n)-175(1-0.99^n)\geq 60$

Let $Y=n(3.01-0.01n)-175(1-0.99^n)$

From GC,

n	Y
53	$59.171<60$
54	$60.084>60$
55	$60.987>60$

Least $n=54$

(ii) Let k be the starting distance between fox and rabbit.

For the fox to never catch up with the rabbit,

$k>\text{Max}\left[n(3.01-0.01n)-175(1-0.99^n)\right]$

To find n for which the fox stop moving, $T_n=0$

$3+(n-1)(-0.02)=0$

$n=151$

The fox takes 150 leaps before it stops moving.

From GC, for $0\leq n\leq 150$,

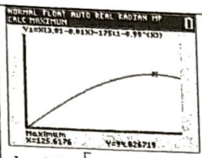
Commented [KSM41]: <u>Question Reading</u>
Question specified that the G.P. is infinite. Many did not read this and proceed to find Sn.
<u>Interpretation</u>
Majority who got this wrong mistook the first term of G.P. to be that of the A.P.

Commented [KSM42]: <u>Strategy</u>
There is no need to deduce the general term from scratch here, and many wasted time to do so.

Commented [KSM43]: Presentation
Students should express this in inequality form to explain why the n obtained is the least.

Commented [KSM44]: Presentation
Those who fail to show either the graph or table will not be awarded full credit.

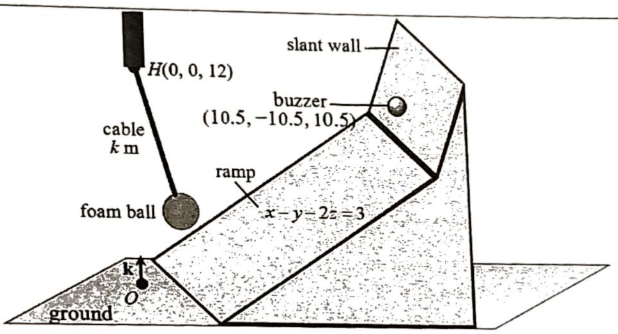
Commented [KSM45]: Interpretation and misconception
Here, at the 151th leap, the fox would have stopped its movement. So it's at the 150th leap before it stopped. Hence, we should analyse the graph of the difference in the distance travelled within the animals' first 150 leaps. Almost all who did this part assumed that the minimum g[ap] between the two animals occurred at the 151t[h] or 150th leap. Do take note the Maximum or minimum point of a graph may not occur at its end points.



$k > \text{Max}\left[n(3.01-0.01n)-175\left(1-0.99^{n}\right)\right]=94.827$

Hence minimum k is 95 m (to the nearest integer)

10 The production team of a popular variety show, *Sprinting Man*, is preparing a site for a segment of the show. In this segment, each participant is to sprint from the starting point, go up a ramp and press a buzzer to complete the challenge.



Referring the starting point as the origin O and the horizontal ground as the x-y plane, the top surface of the ramp has equation $x-y-2z=3$ (see diagram that is not drawn to scale). Distances are measured in metres.

(i) Find the angle of inclination of the ramp. [2]

A spherical polyurethane foam ball of radius 1 m is suspended from a point H with coordinates $(0, 0, 12)$ by a cable of length k m, that is taut all the time. The ball will be swung in various directions during the challenge to increase the level of difficulty.

(ii) If the production team wants to ensure that the foam ball will never come in contact with the ramp, find the range of values that k can take. [3]

The buzzer that the participants are to press is located at the point with coordinates $(10.5, -10.5, 10.5)$. This point lies on a flat slant wall which intersects the ramp along the line l with cartesian equation $x=y+20,\ z=8.5$.

(iii) Find a cartesian equation of the slant wall. [3]

A camera is to be placed along a line L with equation $\mathbf{r}=12\mathbf{k}+t(\mathbf{i}+3\mathbf{j}),\ t\in\mathbb{R}$, with its position denoted by C.

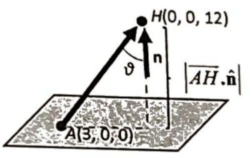
(iv) If the camera is at a distance of $\sqrt{254}$ m from a point P with coordinates $(10, -10, 10)$, determine the possible coordinates of C exactly, showing your workings. Hence deduce the point on L that is nearest to P. [4]

Solution

(i) Angle of inclination of the ramp $= \cos^{-1}\dfrac{\left|\begin{pmatrix}1\\-1\\-2\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right|}{\sqrt{1^2+1^2+2^2}}$

$= \cos^{-1}\dfrac{2}{\sqrt{6}} \approx 35.264^{\circ} = 35.3^{\circ}$ (1 d.p.)

(ii) A point on the ramp is $A(3, 0, 0)$. Let the normal to the ramp be $\mathbf{n} = \begin{pmatrix}1\\-1\\-2\end{pmatrix}$.



Shortest distance from H to ramp

$= |\overrightarrow{AH}.\hat{\mathbf{n}}|$

$= \dfrac{\left|\begin{pmatrix}-3\\0\\12\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\-2\end{pmatrix}\right|}{\sqrt{1+1+4}} = \dfrac{27}{\sqrt{6}}$ (≈ 11.0227)

Since the diameter of the ball is 2 m, $0 < k < \dfrac{27}{\sqrt{6}} - 2$ [or $0 < k < 9.02$ (3 s.f.)].

Commented [TCK46]: Question Reading
The angle of inclination is just the angle between the two planes of the ramp and the ground. Identify the normal of these planes and take the acute angle which means a need to take the modulus value of the dot product.
Many answers were complicated and showed a lack of understanding of the question.
Careless mistake
The dot product and the magnitude of the normal vectors are worked wrongly.

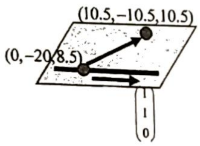
Commented [TCK47]: Interpretation of question
The length of cable plus the diameter of the ball (i.e. k+2) must be less than the shortest distance from H to the ramp if the ball is not to touch the ramp at all.
Presentation
The quality is generally poor with students not explaining what they are doing clearly and not using proper notation, e.g., taking a position vector and cartesian coordinates of a point to be the same, taking x to mean dot product.

(iii) $l: \mathbf{r} = \begin{pmatrix} 0 \\ -20 \\ 8.5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\lambda \in \mathbb{R}$.

A vector parallel to the slant wall is $\begin{pmatrix} 10.5 \\ -10.5 \\ 10.5 \end{pmatrix} - \begin{pmatrix} 0 \\ -20 \\ 8.5 \end{pmatrix} = \begin{pmatrix} 10.5 \\ 9.5 \\ 2 \end{pmatrix}$.

Therefore, a normal to the slant wall is

$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 10.5 \\ 9.5 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$.



$\mathbf{r} \cdot \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 10.5 \\ -10.5 \\ 10.5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} = 31.5$

A cartesian equation of the slant wall: $4x - 4y - 2z = 63$.

Commented [TCK48]: Presentation

Many still write $l = \begin{pmatrix} 0 \\ -20 \\ 8.5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and not explain what λ is.

Careless mistake

Many do not convert the equation of l from cartesian to vector form successfully. This affects all subsequent working.

Commented [TCK49]: Interpretation of question

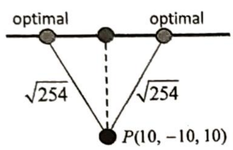
This question is about finding the equation of the plane of the slant wall. To find it, we need a point on the plane and the normal vector. We have the point which is (10.5, -10.5, 10.5) but not the normal vector. To find the normal, we need two direction vectors parallel to the plane.

Misconception

Many students took the normal vector of the ramp as one of the direction vectors.

(iv) Camera lies along the line with equation $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 12 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, t \in \mathbb{R}$.

As the camera needs to be $\sqrt{254}$ m from P,



$$\sqrt{(t-10)^2 + (3t+10)^2 + (12-10)^2} = \sqrt{254}$$

$$\left(t^2 - 20t + 100\right) + \left(9t^2 + 60t + 100\right) + 4 = 254$$

$$10t^2 + 40t - 50 = 0$$

$$(t+5)(t-1) = 0$$

$$\therefore t = 1 \text{ or } t = -5.$$

The corresponding optimal positions are

(1, 3, 12) and (−5, −15, 12).

By symmetry, the point along the line closest to P is the **midpoint** of the two optimal positions. Therefore, this point has coordinates

$$\left(\frac{1-5}{2}, \frac{3-15}{2}, 12\right) = (-2, -6, 12).$$

Commented [TCK50]: Careless mistake
Taking $12\underset{\sim}{k}$ as $\begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix}$.

Commented [TCK51]: Careless mistake
Errors in expansion. Students should perhaps use the GC to solve as a safeguard.

Commented [TCK52]: Presentation
Not giving in coordinates form but as column vectors instead.

Commented [TCK53]: Presentation
Instead of mid-point theorem, many solved from scratch for the point on L nearest to P.

11	The cylindrical tank in a research laboratory has a cross-sectional area of 4 m^2. To cool the tank, water is pumped in and out of the tank simultaneously. The volume and height of the water in the tank at any time t minutes is given by V (litres) and h (metres) respectively. Clean water is pumped into the tank at a rate that is proportional to h^2 and the water is pumped out from the tank at a rate that is proportional to h.	
	(i) Assume that the water does not overflow and that there is no change to the height of the water when h is 10, show that $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{kh(h-10)}{4}$ where k is a real constant.	[4]
	The tank was initially filled with clean water to a height of 2 metres. When the height of the water is 5 metres, the volume of water is increasing at a rate of 5.5 litres per minute.	
	(ii) Find the exact value of k. Hence find h in terms of t.	[5]
	(iii) Sketch a graph of h against t. Hence write down the minimum height of the cylindrical tank that will not result in the overflow of the water.	[3]
	Solution	
	(i) $\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\mathrm{d}V_{\text{in}}}{\mathrm{d}t_{\text{in}}}-\frac{\mathrm{d}V_{\text{out}}}{\mathrm{d}t_{\text{out}}}$ $\frac{\mathrm{d}V}{\mathrm{d}t}=Ah^2-Bh, \quad A,B\in\mathbb{R}$	
	When $h=10$, $\frac{\mathrm{d}V}{\mathrm{d}t}=0$.	
	$B=10A$	
	Since $V=\pi r^2h$ (and given that base area is 4 m^2) $\therefore V=4h$ $\frac{\mathrm{d}V}{\mathrm{d}t}=4\frac{\mathrm{d}h}{\mathrm{d}t}$	
	$\Rightarrow 4\frac{\mathrm{d}h}{\mathrm{d}t}=Ah^2-10Ah$	
	$\Rightarrow \frac{\mathrm{d}h}{\mathrm{d}t}=\frac{kh(h-10)}{4}$, where $A=k$	
	(ii) $\frac{\mathrm{d}V}{\mathrm{d}t}=5.5$ $5.5=\frac{5k(5-10)}{4}\times 4$ $k=-\frac{11}{50}$	
	$\frac{\mathrm{d}h}{\mathrm{d}t}=-\frac{11h(h-10)}{200}$ $\int\frac{1}{h^2-10h}\,\mathrm{d}h=-\int\frac{11}{200}\,\mathrm{d}t$	

Commented [SH54]: 1.<u>Question reading</u>

"Clean water is pumped into the tank at a rate that is proportional to h^2 and the water is pumped out from the tank at a rate that is proportional to h."

The above para – refers to Vol of water per unit time. Thus, the derivative $\frac{\mathrm{d}V}{\mathrm{d}t}$ should be used.

The proportionality of constant must be different for the rate of Clean water pumped in and for the rate of water pumped out. Majority of students used the same constant.

2.<u>Presentation / Conceptual understanding</u>

$\frac{\mathrm{d}h}{\mathrm{d}t}=Ah^2-Bh$ (Incorrect).

$\frac{\mathrm{d}h}{\mathrm{d}t}=0, h=10, h(Ah-B)=0$. Students used this derivative to calculate the value of B which is incorrect.

If Students are using the below expression to solve for B, then it is correct.

$$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\frac{\mathrm{d}V}{\mathrm{d}t}}{\frac{\mathrm{d}V}{\mathrm{d}h}}=\frac{Ah^2-Bh}{4}$$

3.It should be $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{h=10}=Ah^2-Bh=0$

$$\left.\frac{\mathrm{d}h}{\mathrm{d}t}\right|_{h=10}=\frac{Ah^2-Bh}{4}=0$$

Commented [SH55]: <u>Presentation</u>

Final answer must be shown as given in the question.

$$\int \frac{1}{(h-5)^2-5^2}\,dh = -\frac{11}{200}t + c$$

$$\frac{1}{10}\ln\left|\frac{(h-5)-5}{h}\right| = -\frac{11}{200}t + c$$

$$\ln\left|\frac{h-10}{h}\right| = -\frac{11}{20}t + 10c$$

$$\left|1-\frac{10}{h}\right| = e^{-\frac{11}{20}t+10c}$$

$$1-\frac{10}{h} = \pm e^{-\frac{11}{20}t+10c}$$

$$\frac{10}{h} = 1 + Ae^{-\frac{11}{20}t} \qquad A = \pm e^{10c}$$

$$h = \frac{10}{1+Ae^{-\frac{11}{20}t}}$$

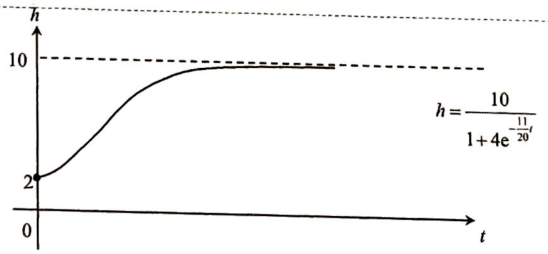
When $t = 0,\ h = 2$

$$2 = \frac{10}{1+A}$$

$A = 4$

$$h = \frac{10}{1+4e^{-\frac{11}{20}t}}$$

(iii)



Minimum height of the cylindrical tank = 10 metres

Commented [SH56]: 1.Conceptual mistake
Many students, did not use the correct technique to integrate using Variable separable method.

Method to solve : By Partial Fractions
Or by Completing the Square

Commented [SH57]: Presentation

1.Majority of students could not simplify $\frac{h-10}{h}$ to $1-\frac{10}{h}$. If this was done earlier, then the final answer will be neat and easy to work with.

2. ln (function) is defined only when the function is positive. In circumstances that you are unsure, its always safe to place the modulus. So for this question, many students omitted the modulus.

3. Removal of Modulus sign must follow through in the presentation of the answer. Many neglected this working and was penalized for not showing the removal of modulus which will manifest $\pm$ in the subsequent line.

Commented [SH58]: Presentation of Graphs

1.Initial height = 2m must be shown on the graph

2.Sketch the graph $h = \frac{10}{1+4e^{-\frac{11}{20}t}}$ to check the shape of the curve. Graph should only be drawn on the positive axis due to the context of the question.

3.Height = 10 must be captured as horizontal asymptote and graph sketched must not touch the asymptote

4. Many overlooked and lost 1 mark for not stating/ writing the min. height such that the water will not overflow.

[Turn Over