Question 1 (Power Series)

(i) $f'(x) = f(x)\ln(\sec ax) \quad (1)$

Differentiating (1) both sides implicitly wrt x,

$$f''(x) = f'(x)\ln(\sec ax) + f(x)\left(\frac{a\sec ax\tan ax}{\sec ax}\right)$$

$$f''(x) = f'(x)\left[\frac{f'(x)}{f(x)}\right] + f(x)\left(a\tan ax\right)$$

$$f''(x)f(x) = \left[f'(x)\right]^2 + a\left[f(x)\right]^2\tan(ax)\,(\text{shown})$$

$$f'''(x)f(x) + f''(x)f'(x)$$
$$= 2f'(x)f''(x) + \left[f(x)\right]^2 a^2\sec^2 ax$$
$$+ 2a\,f(x)f'(x)\tan ax$$

$$f'''(x)f(x) - f''(x)f'(x)$$
$$= \left[f(x)\right]^2 a^2\sec^2 ax + 2a\,f(x)f'(x)\tan ax$$

$$f(0) = \frac{1}{2},\ f'(0) = 0,\ f''(0) = 0,\ f'''(0) = \frac{1}{2}a^2$$

$$\therefore f(x) = \frac{1}{2} + \frac{\left(\frac{1}{2}a^2\right)}{3!}x^3 + \cdots = \frac{1}{2} + \frac{1}{12}a^2x^3 + \cdots$$

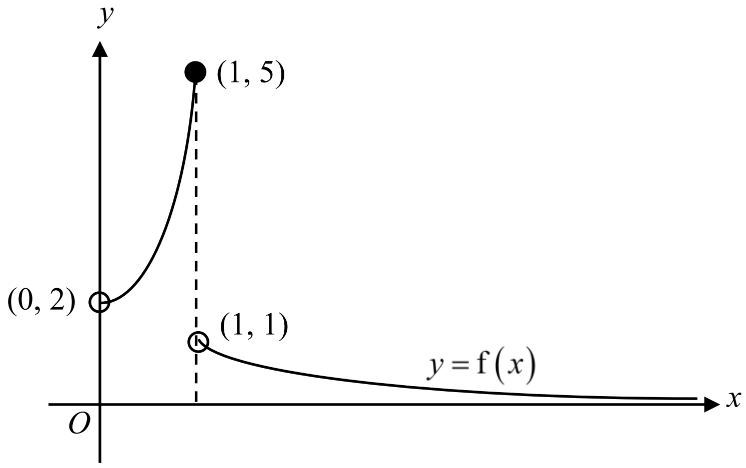
(ii) $f(x)\left[1+\ln(\cos 3x)\right] = f(x)\left[1-\ln(\sec 3x)\right]$

$$= f(x) - f(x)\ln(\sec 3x)$$
$$= f(x) - f'(x)$$
$$= \frac{1}{2} - \frac{1}{4}(3)^2x^2 + \cdots$$
$$= \frac{1}{2} - \frac{9}{4}x^2 + \cdots$$

$$g(x) = \int \frac{1}{2} - \frac{9}{4}x^2 + \cdots \,dx$$
$$= c + \frac{1}{2}x - \frac{3}{4}x^3 + \cdots$$

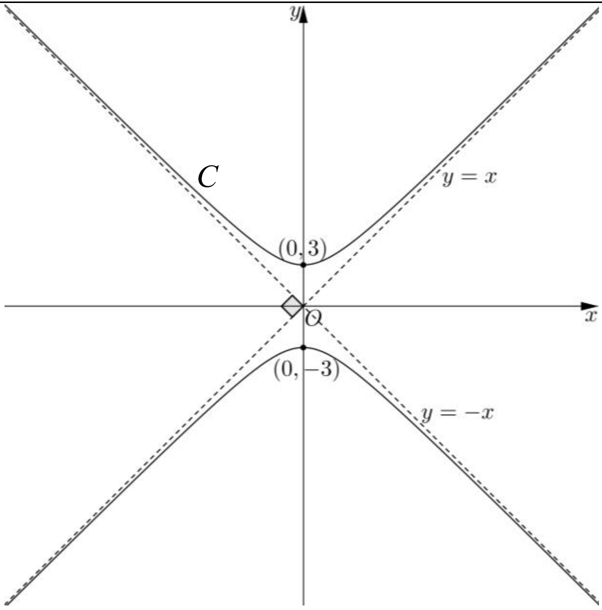
Since $g(0) = 0,\ c = 0.\ \therefore g(x) = \frac{1}{2}x - \frac{3}{4}x^3 + \cdots$

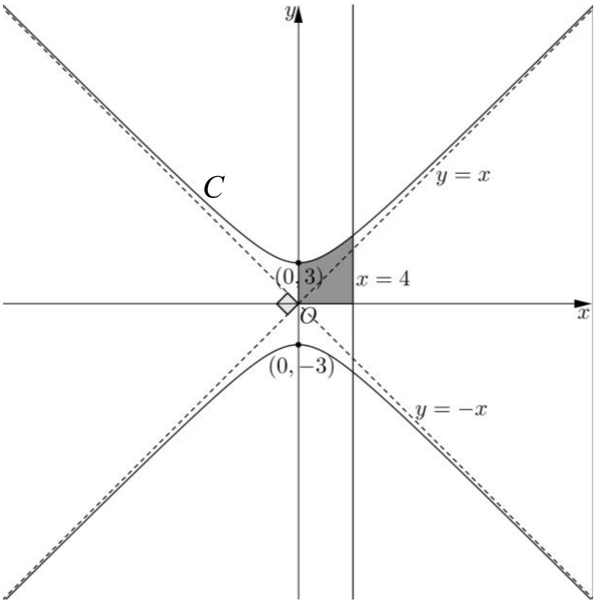
Question 2 (Functions)

(i) 	
(ii) $\mathrm{D_f} = (0,\infty),\ \mathrm{R_f} = (0,1)\cup(2,5]$ Since $\mathrm{R_f} \subseteq \mathrm{D_f}$, If $0 < x \le 1$, $\mathrm{f}^2(x) = \mathrm{f}\left(3x^2+2\right) = \dfrac{1}{3x^2+2}$ If $x > 1$, $\mathrm{f}^2(x) = \mathrm{f}\left(\dfrac{1}{x}\right) = 3\left(\dfrac{1}{x}\right)^2 + 2 = \dfrac{3}{x^2}+2$ $\therefore \mathrm{f}^2(x) = \begin{cases} \dfrac{1}{3x^2+2}, & \text{for } x\in\mathbb{R},\ 0<x\le 1, \\ \dfrac{3}{x^2}+2, & \text{for } x\in\mathbb{R},\ x>1. \end{cases}$	

(iii) From the graph in part **(i)**, we see that for $2 < y \leq 5$, $y = \mathrm{f}(x) = 3x^2 + 2$ $3x^2 = y - 2$ $x^2 = \dfrac{y-2}{3}$ $x = \sqrt{\dfrac{y-2}{3}} \quad (\because x > 0)$ and for $0 < y < 1$, $y = \mathrm{f}(x) = \dfrac{1}{x} \Rightarrow x = \dfrac{1}{y}$ Thus, $\mathrm{f}^{-1}(x) = \begin{cases} \dfrac{1}{x}, & \text{for } x \in \mathbb{R}, 0 < x < 1, \\ \sqrt{\dfrac{x-2}{3}}, & \text{for } x \in \mathbb{R}, 2 < x \leq 5. \end{cases}$	

Question 3 (Conics, Area, Integration by Substitution)

(i) 	
(ii) $\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sec\theta\tan\theta\right) = \sec\theta\tan\theta\cdot\tan\theta + \sec\theta\cdot\sec^2\theta$ $= \sec\theta\tan^2\theta + \sec^3\theta$ $= \sec\theta\left(\sec^2\theta - 1\right) + \sec^3\theta$ $= 2\sec^3\theta - \sec\theta$ (shown)	

(iii) 	

$$\text{Area} = \int_0^4 y \, dx$$

$$= \int_0^4 \sqrt{x^2+9} \, dx$$

$$= \int_0^{\tan^{-1}\frac{4}{3}} \sqrt{9\tan^2\theta + 9}\left(3\sec^2\theta\right) d\theta$$

$$= \int_0^{\tan^{-1}\frac{4}{3}} 3\sec\theta \cdot 3\sec^2\theta \, d\theta$$

$$= 9\int_0^{\tan^{-1}\frac{4}{3}} \sec^3\theta \, d\theta$$

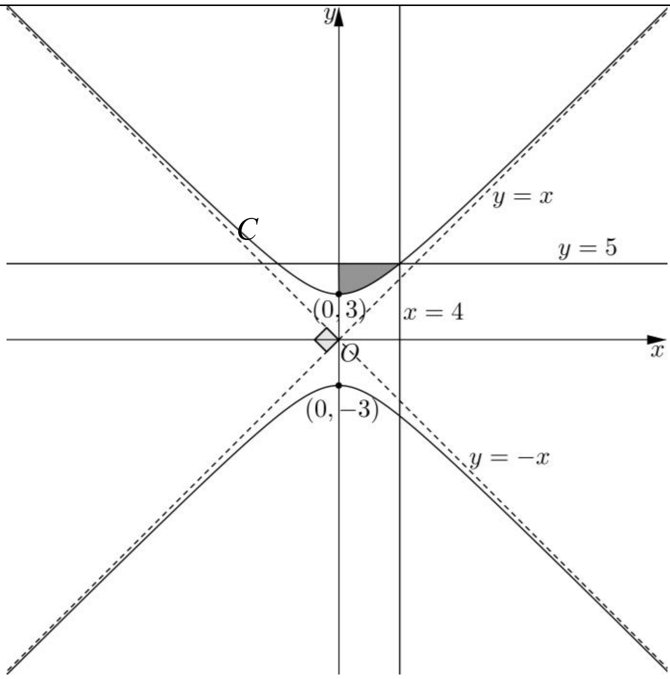
$$= \frac{9}{2}\int_0^{\tan^{-1}\frac{4}{3}} \left(2\sec^3\theta - \sec\theta\right) + \sec\theta \, d\theta$$

$$= \frac{9}{2}\Big[\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\Big]_0^{\tan^{-1}\frac{4}{3}}$$

$$= \frac{9}{2}\left[\left(\frac{\sqrt{4^2+3^2}}{3}\right)\left(\frac{4}{3}\right) + \ln\left|\frac{\sqrt{4^2+3^2}}{3} + \frac{4}{3}\right|\right]$$

$$= \frac{9}{2}\left(\frac{20}{9} + \ln 3\right)$$

$$= 10 + \frac{9}{2}\ln 3 \text{ units}^2$$

(iv)  $\int_3^5 \sqrt{x^2-9}\ \mathrm{d}x = \int_3^5 \sqrt{y^2-9}\ \mathrm{d}y$ $= 4\cdot 5 - \text{area of } S$ $= 20 - \left(10 + \frac{9}{2}\ln 3\right)$ $= 10 - \frac{9}{2}\ln 3$	

Question 4 (Curve Sketching)

(i) $y=\frac{a^2}{x}-\frac{1}{x-k}\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{a^2}{x^2}+\frac{1}{(x-k)^2}$ At the stationary points, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$. Thus, $-\frac{a^2}{x^2}+\frac{1}{(x-k)^2}=0$ $\frac{1}{(x-k)^2}=\frac{a^2}{x^2}$ $\left(\frac{x}{x-k}\right)^2=a^2$ $\frac{x}{x-k}=\pm a$ $\frac{x}{x-k}=a$ or $\frac{x}{x-k}=-a$ $x=ax-ak$ or $x=-ax+ak$ $(a-1)x=ak$ or $(a+1)x=ak$ $x=\frac{ak}{a-1}$ or $x=\frac{ak}{a+1}$	

(ii) The larger x-coordinate of the two stationary points is $x = \dfrac{ak}{a-1}$.

The y-coordinate of this point is

$$y = \frac{a^2}{\left(\dfrac{ak}{a-1}\right)} - \frac{1}{\dfrac{ak}{a-1} - k}$$

$$= \frac{a(a-1)}{k} - \frac{a-1}{ak - k(a-1)}$$

$$= \frac{a^2 - a}{k} - \frac{a-1}{k}$$

$$= \frac{a^2 - 2a + 1}{k}$$

$$= \frac{(a-1)^2}{k}$$

So the locus of P has parametric equations

$$\begin{cases} x = \dfrac{ak}{a-1} & (1) \\ y = \dfrac{(a-1)^2}{k} & (2) \end{cases}$$

From (1), $x = \dfrac{ak}{a-1} \Rightarrow ax - x = ak$

$$\Rightarrow ax - ak = x$$

$$\Rightarrow a(x-k) = x$$

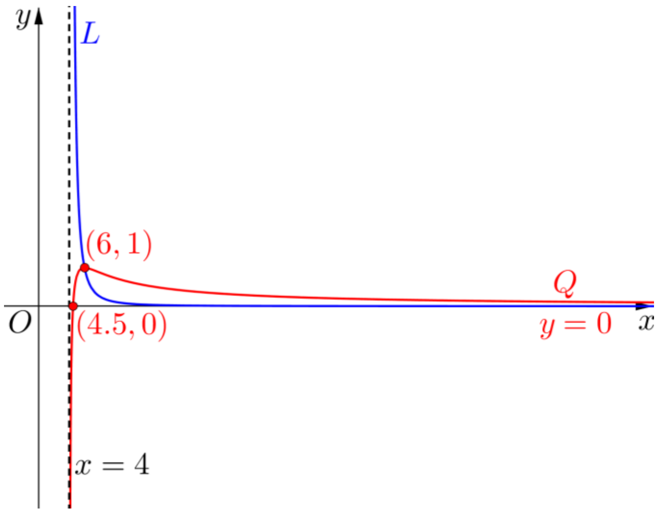
$$\Rightarrow a = \frac{x}{x-k} \qquad (3)$$

Substituting (3) into (2), $y = \dfrac{(a-1)^2}{k}$

$$y = \frac{\left(\dfrac{x}{x-k} - 1\right)^2}{k}$$

$$y = \frac{\left(\dfrac{x - x + k}{x-k}\right)^2}{k}$$

$$y = \frac{k}{(x-k)^2}$$

(iii) 	

Question 5 (Binomial Distribution – Mode)

$W \sim \mathrm{B}(n, p)$ $\mathrm{P}(W = m) \geq \mathrm{P}(W = m+1)$ $\Rightarrow \binom{n}{m}(p)^m(1-p)^{n-m} \geq \binom{n}{m+1}(p)^{m+1}(1-p)^{n-m-1}$ $\Rightarrow \frac{n!}{m!(n-m)!}(1-p) \geq \frac{n!}{(m+1)!(n-m-1)!}(p)$ $\Rightarrow (m+1)!(n-m-1)!(1-p) \geq m!(n-m)!(p)$ $\Rightarrow (1-p)(m+1) \geq p(n-m)$ $\Rightarrow m+1-mp-p \geq np-mp$ $\Rightarrow m \geq np+p-1$ $\Rightarrow m \geq \mathrm{E}(W)+p-1$ (since $\mathrm{E}(W)=np$)(shown)	

Question 6 (Normal Distribution)

(i)	Let X be the the waiting time for a randomly chosen bus 55 on a weekday. $X \sim \mathrm{N}\left(9, \frac{1.5^2}{50}\right)$ Let $\overline{X} = \frac{X_1 + X_2 + \cdots + X_{50}}{50} \sim \mathrm{N}\left(9, \frac{1.5^2}{50}\right)$ $\mathrm{P}(\overline{X} > 9.3) = 0.0786$	
(ii)	Since the number of days, 50, is large, the sample mean waiting time follows a normal distribution approximately by Central Limit Theorem. Therefore the calculations in part **(i)** will still be valid.	
(iii)	Let Y be the journey time (in minutes) to the MRT station for a randomly chosen bus on a weekday, and let t be the time taken (in minutes) such that she misses the train less than 30% of the time. $Y \sim \mathrm{N}\left(10,\ 1.1^2\right)$ $X + Y \sim \mathrm{N}\left(19, 3.46\right)$ $\mathrm{P}(X + Y > t) < 0.3$ $t > 19.975$ Least $t = 20$ mins Latest time to leave home is 6.40 am	
(iv)	Let $W = 1.3X_1 - X_2 \sim \mathrm{N}\left(2.7, 1.3^2\left(1.5^2\right) + 1.5^2\right)$ $\mathrm{P}(-2 \le W \le 2) = 0.3599 \approx 0.360$	

Question 7 (Binomial Distribution – Finding Probabilities)

(i) In this context, a random sample is one such that every marker produced has equal probability of being selected in this sample and the markers are selected independently of one another.

(ii) $X \sim \mathrm{B}(n, 0.078)$

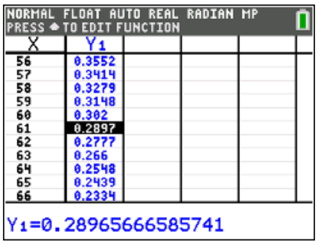
If $n - X$ markers are not faulty, X markers are faulty. Thus,

$$\mathrm{P}(n - X > n - 4) < 0.3$$

$$\mathrm{P}(X < 4) < 0.3$$

$$\mathrm{P}(X \le 3) < 0.3$$

Using GC and from the table,



Least value of n is 61.

(iii) $X \sim \mathrm{B}(10, 0.078)$

$\mathrm{P}(X = 0) = 0.4439246$

$\mathrm{P}(X \le 2) = 0.962450$

$\mathrm{P}(1 \le X \le 2) = \mathrm{P}(X \le 2) - \mathrm{P}(X = 0) = 0.518526$

Thus, the required probability is

$$\frac{\binom{50}{35}\left(\mathrm{P}(1 \le X \le 2)\right)^{35}\left(\mathrm{P}(X = 0)\right)^{15}}{\left(\mathrm{P}(X \le 2)\right)^{50}}$$

$$= \frac{\binom{50}{35}(0.518526)^{35}(0.4439246)^{15}}{(0.962450)^{50}}$$

$= 0.0081291$

$= 0.00813$ (to 3 s.f.)

Alternatively, let Y be the number of samples with at least one faulty marker, out of 50 samples with at most 2 faulty markers each. Then Y is binomially distributed with probability of success

$$p' = \frac{\mathrm{P}(1 \le X \le 2)}{\mathrm{P}(X \le 2)} = \frac{0.518526}{0.962450} = 0.538756$$

Thus, $Y \sim \mathrm{B}(50, 0.538756)$. So,

$\mathrm{P}(Y = 35) = 0.0081288$

$= 0.00813$ (to 3 s.f.)

Question 8 (Discrete Random Variables)

(i) Since sum of all probabilities = 1, $\frac{32}{81}+a+\frac{8}{27}+b+c=1\Rightarrow a+b+c=1-\frac{32}{81}-\frac{8}{27}=\frac{25}{81}$ --- (1) Since $\mathrm{E}\left(\lvert Y-3\rvert\right)=\frac{232}{81}$, $\lvert -1-3\rvert\frac{32}{81}+\lvert 0-3\rvert a+\lvert 1-3\rvert\frac{8}{27}+\lvert 2-3\rvert b+\lvert 3-3\rvert c$ $=4\left(\frac{32}{81}\right)+3a+2\left(\frac{8}{27}\right)+b$ $4\left(\frac{32}{81}\right)+3a+2\left(\frac{8}{27}\right)+b=\frac{232}{81}\Rightarrow 3a+b=\frac{56}{81}$ --- (2) $3\times(1)-(2):\ 3(a+b+c)-(3a+b)=3\times\frac{25}{81}-\frac{56}{81}$ $2b+3c=\frac{19}{81}$ (shown) --- (3) **Alternatively**, we observe that the possible values of Y are all not more than 3. Thus, $\mathrm{E}\left(\lvert Y-3\rvert\right)=\frac{232}{81}\Rightarrow \mathrm{E}(3-Y)=\frac{232}{81}\Rightarrow 3-\mathrm{E}(Y)=\frac{232}{81}$ $\Rightarrow \mathrm{E}(Y)=3-\frac{232}{81}=\frac{11}{81}$ Hence, we have $(-1)\times\frac{32}{81}+0\times a+1\times\frac{8}{27}+2b+3c=\frac{11}{81}$ $-\frac{8}{81}+2b+3c=\frac{11}{81}$ $2b+3c=\frac{19}{81}$ (shown) --- (3)	

(ii) $\mathrm{E}(Y) = -\frac{32}{81} + \frac{8}{27} + 2b + 3c$ $= -\frac{8}{81} + 2b + 3c$ $= -\frac{8}{81} + \frac{19}{81}$ (from **(i)**) $= \frac{11}{81}$ Since $\mathrm{Var}(Y) = \frac{7736}{6561}$, $\mathrm{E}(Y^2) - (\mathrm{E}(Y))^2 = \frac{7736}{6561}$ $\frac{32}{81} + \frac{8}{27} + 4b + 9c - \left(\frac{11}{81}\right)^2 = \frac{7736}{6561}$ $4b + 9c = \frac{7736}{6561} - \frac{56}{81} + \left(\frac{11}{81}\right)^2$ $4b + 9c = \frac{41}{81}$ --- (4) By using the Simultaneous Equation Solver in the GC to solve equations (1), (3) and (4) simultaneously, $a = \frac{16}{81},\ b = \frac{8}{81},\ c = \frac{1}{81}$	
(iii) *Y* does not follow a binomial distribution since it can possibly take a value of -1 which is not permissible in a binomial distribution that counts the number of success.	

Question 9 (Hypothesis Testing)

(i) Let $x = t - 20$. Then $\sum x = 215;\quad \sum x^2 = 3234.$

$\bar{x} = \bar{t} - 20$

$\Rightarrow \bar{t} = \bar{x} + 20$

$\Rightarrow \bar{t} = \frac{215}{72} + 20 = \frac{1655}{72} = 22.986 = 23.0$ (to 3 s.f.)

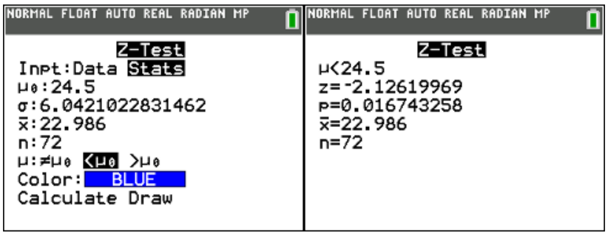
$s^2 = \frac{1}{71}\left(3234 - \frac{215^2}{72}\right) = 36.507 = 36.5$ (to 3 s.f.)

(ii) Let μ be Osoi's population mean travelling time (in minutes) from home to school on a randomly chosen morning after he started taking the alternative route

To test $\mathrm{H}_0 : \mu = 24.5$ against $\mathrm{H}_1 : \mu < 24.5$

Level of significance, $\alpha = 0.05$ (lower-tailed)

Under H_0, $Z = \dfrac{\bar{T} - 24.5}{\sqrt{\dfrac{36.507}{72}}} \sim \mathrm{N}(0,1)$ approximately by Central Limit Theorem, since $n = 72$ is large (> 30).

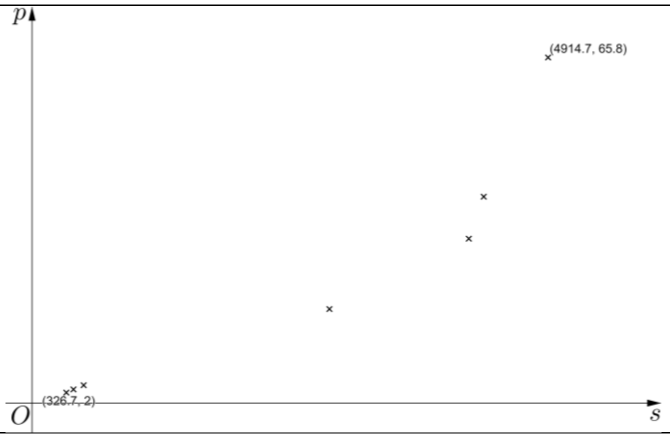
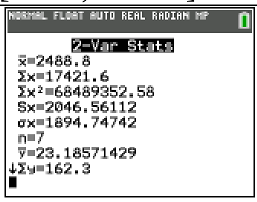


Method 1: *p*-value	Method 2: test statistic
p-value $= 0.0167$ $\alpha = 0.05$ $\therefore p < \alpha$	Test statistic, $z_{\text{calc}} = -2.13$ Critical value, $z_{\text{crit}} = z_{0.05} = -1.645$ $\therefore z_{\text{calc}} < z_{\text{crit}}$

Thus, we reject H_0. We conclude that there is **sufficient evidence** at the **5% level of significance** to claim that **Osoi's population mean travelling time from home to school in the morning has shortened after taking the alternative route to school.**

(iii)	Test $H_0 : \mu = 24.5$ against $H_1 : \mu \neq 24.5$ Level of significance, $\alpha = 0.05$ (two-tailed) Standardised critical region: $z \leq -1.960$ or $z \geq 1.960$ Under H_0, $Z = \dfrac{\overline{T} - 24.5}{\sqrt{\dfrac{36.507}{72}}} \sim N(0,1)$ approximately by Central Limit Theorem, since n = 72 is large (> 30). The observed test statistic value is $\dfrac{\overline{t} - 24.5}{\sqrt{\dfrac{36.507}{72}}}$. Thus the critical region for this test is given by $\left\| \dfrac{\overline{t} - 24.5}{\sqrt{\dfrac{36.507}{72}}} \right\| \geq 1.960$ $\left\| \overline{t} - 24.5 \right\| \geq 1.960\sqrt{\dfrac{36.507}{72}}$ $\overline{t} \leq 24.5 - 1.960\sqrt{\dfrac{36.507}{72}}$ or $\overline{t} \geq 24.5 + 1.960\sqrt{\dfrac{36.507}{72}}$ $\overline{t} \leq 23.1$ or $\overline{t} \geq 25.9$	

Question 10 (Correlation and Regression)

(i)	 	
(ii) (a)	The product moment correlation coefficient between *s* and *p* = 0.928578571 = 0.929 (3 s.f)	
(ii) (b)	The product moment correlation coefficient between s^2 and *p* = 0.970297308 = 0.970 (3 s.f.)	
(iii)	The scatter plot appears to follow a curve that is increasing with an increasing gradient (concave upwards). Also, the *r*-value for model II, 0.970, has an **absolute** value that is closer to 1 than the *r*-value for model I, 0.929. Thus, Model II is better.	
(iv)	Regression line of *p* on s^2: $p=\left(2.284063863\times10^{-6}\right)s^2+0.8379921133$ $p=\left(2.28\times10^{-6}\right)s^2+0.838$	
(v)	For $s=300$, $p=\left(2.284063863\times10^{-6}\right)(300)^2+0.8379921133$ $=1.043557858$ $=1.04$ (to 3s.f.) The estimate is not reliable as the given value of *s*, 300, does not lie in the data range for *s*, [326.7, 4914.7]	
(vi)	*p* = 0.010423445*s*– 4.043538369  $\bar{s}=\dfrac{17421.6+3689}{8}=2638.825$ $(\bar{s},\,\bar{p})$ lies on the regression line: $\bar{p}=0.010423445(2638.825)-4.043538369$ $=23.46210888$ Thus, missing *p* = 23.46210888 × 8 – 162.3 = 25.39687107 = 25.4 (3sf)	

Question 11 (Permutations and Combinations, Probability)

(i)	$\mathrm{P}\left(\text{selections are all distinct}\right)$ $= \dfrac{\left[\binom{5}{2}\times 2!\right]\times\left[\binom{16}{3}\times 3!\right]}{5^2\times 16^3}$ or $\dfrac{5\times 4}{5^2}\times\dfrac{16\times 15\times 14}{16^3}$ $=\dfrac{21}{32}$	
(ii)	$\mathrm{P}\left(\text{girl in 1}^{\text{st}}\text{ round, boy in 2}^{\text{nd}}\text{ round}\right)$ $=\dfrac{\binom{3}{1}\binom{2}{1}}{5^2}$ $=\dfrac{6}{25}$	
(iii)	$\mathrm{P}(A)=\dfrac{\binom{5}{1}}{5^2}=\dfrac{1}{5}$ $\mathrm{P}(B)=\dfrac{\binom{16}{1}\binom{15}{1}\times\frac{3!}{2!}}{16^3}=\dfrac{45}{256}$ Since the outcomes of the selections of the OGLs and the orientees bear no influence on each other, the two events are independent. So, $\mathrm{P}(A\cap B)=\mathrm{P}(A)\times\mathrm{P}(B)$ $\mathrm{P}\left(A\text{ occurs or }B\text{ occurs but not both}\right)$ $=\mathrm{P}(A\cup B)-\mathrm{P}(A\cap B)$ $=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)-\mathrm{P}(A\cap B)$ $=\mathrm{P}(A)+\mathrm{P}(B)-2\mathrm{P}(A\cap B)$ $=\mathrm{P}(A)+\mathrm{P}(B)-2\mathrm{P}(A)\times\mathrm{P}(B)$ $=\dfrac{1}{5}+\dfrac{45}{256}-2\left(\dfrac{1}{5}\right)\left(\dfrac{45}{256}\right)$ $=\dfrac{391}{1280}$	

	Alternatively, P(exactly one OGL and exactly 1 or 3 orientees) $=\frac{1}{5}\left[\left(\frac{1}{16}\right)^2+\frac{15}{16}\times\frac{14}{16}\right]=\frac{211}{1280}$ P(two OGLs and exactly 2 orientees) $=\frac{4}{5}\binom{3}{1}\left(\frac{1}{16}\times\frac{15}{16}\right)=\frac{9}{64}$ Required Probability $=\frac{211}{1280}+\frac{9}{64}=\frac{391}{1280}$	
(iv)	P(Chairperson selected exactly once) $=\frac{1\times15^2\times\binom{3}{1}}{16^3}$ $=\frac{675}{4096}$	
(v)	P(all girls | Chairperson selected exactly once) $=\frac{n(\text{all girls, Chairperson selected exactly once})}{n(\text{Chairperson selected exactly once})}$ $=\frac{1\times9^2\times\binom{3}{1}\times3^2}{1\times15^2\times\binom{3}{1}\times5^2}$ $=\frac{81}{625}$	