2020 RI H2 Mathematics Prelim Paper 2 Solutions

Section A: Pure Mathematics [40 marks]

1 **[5]**	Given that $u_1 = \frac{3}{2}$, let the common difference of the arithmetic series be *d* and the common ratio of the geometric series be *r*. and $\frac{3}{2}+\frac{3}{2}r+\frac{3}{2}r^2=\frac{21}{2}$ −−−−−−−−(1) $r^2+r-6=0$ $(r+3)(r-2)=0$ $\therefore r=2,$ since r is non negative. $\frac{3}{2}+3d=\frac{3}{2}r^3$ -----------------(2) Substitute *r* = 2 into (2), $d=\frac{7}{2}$. Sum of first 10 odd numbered terms of AP $S=\frac{10}{2}\left[2\left(\frac{3}{2}\right)+9(7)\right]=330$

2 **[5]**	$S_4=\frac{a(r^4-1)}{r-1}$ −−−−−−−−−(1) $S_8=\frac{a(r^8-1)}{r-1}$ −−−−−−−−−(2) $\frac{(2)}{(1)} \Rightarrow \frac{r^8-1}{r^4-1}=\frac{17}{16}$ $\frac{\left(r^4-1\right)\left(r^4+1\right)}{r^4-1}=\frac{17}{16}$ $\therefore r^4=\frac{1}{16} \Rightarrow r_1=\frac{1}{2},\ r_2=-\frac{1}{2}$ When $r_1=\frac{1}{2},\ S_1=\frac{a}{1-\frac{1}{2}}=2a$ When $r_2=-\frac{1}{2},\ S_2=\frac{b}{1+\frac{1}{2}}=\frac{2}{3}b$ Ratio is $a:\frac{1}{3}b$ or 3*a* : *b*

3(i) **[4]**	$R_g = (-\infty, 3)$ $D_f = (0, \infty)$ Since $R_g \not\subset D_f$, fg does not exist. $R_f = (1, 4)$ $D_g = (-\infty, 4)$ Since $R_f \subseteq D_g$, gf exists. $R_{gf} = [g(2), g(4))$ $= [-1, 3)$
(ii) **[4]**	Largest possible value of c is 1. For $x < 1$, $\|x-1\| = 1-x$ $g(x) = (1-x)(x-3)$ $(1-x)(x-3) = y$ $1-(x-2)^2 = y$ $x = 2-\sqrt{1-y} \quad \because x<1$ $g^{-1}(x) = 2-\sqrt{1-x},\ x \in \mathbb{R},\ x<0.$

4 **[2]**	$y = e^{\tan^{-1}\left(\frac{x}{2}\right)}$ $\frac{dy}{dx} = e^{\tan^{-1}\left(\frac{x}{2}\right)} \frac{1}{1+\left(\frac{x}{2}\right)^2} \times \frac{1}{2}$ $= \frac{2e^{\tan^{-1}\left(\frac{x}{2}\right)}}{4+x^2}$ $\left(4+x^2\right)\frac{dy}{dx} = 2e^{\tan^{-1}\left(\frac{x}{2}\right)}$ $\left(4+x^2\right)\frac{dy}{dx} = 2y \quad$ (Shown)

4 **(i)** **[5]**	$\left(4+x^2\right)\dfrac{\mathrm{d}y}{\mathrm{d}x}=2y$ ---------- (1) $\left(4+x^2\right)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+2x\dfrac{\mathrm{d}y}{\mathrm{d}x}=2\dfrac{\mathrm{d}y}{\mathrm{d}x}$ ---------- (2) $\left(4+x^2\right)\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}+2x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+2x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\dfrac{\mathrm{d}y}{\mathrm{d}x}=2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ $\left(4+x^2\right)\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}+4x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\dfrac{\mathrm{d}y}{\mathrm{d}x}=2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ ---------- (3) When $x=0,\ y=1$. From equations (1), (2), (3) we get $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{2},\ \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\dfrac{1}{4},\ \dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}=-\dfrac{1}{8}$. By Maclaurin's Theorem, $\mathrm{e}^{\tan^{-1}\left(\frac{x}{2}\right)}=1+\dfrac{1}{2}x+\dfrac{1}{4}\left(\dfrac{x^2}{2!}\right)-\dfrac{1}{8}\left(\dfrac{x^3}{3!}\right)$ $=1+\dfrac{1}{2}x+\dfrac{1}{8}x^2-\dfrac{1}{48}x^3+\dots$
(ii) **[3]**	$\mathrm{e}^{\tan^{-1}\left(\frac{x}{2}\right)}=1+\dfrac{1}{2}x+\dfrac{1}{8}x^2-\dfrac{1}{48}x^3+\dots$ $\dfrac{\mathrm{e}^{\tan^{-1}\left(\frac{x}{2}\right)}}{\left(1+x\right)^2}=\left(1+\dfrac{1}{2}x+\dfrac{1}{8}x^2-\dots\right)\left(1+x\right)^{-2}$ $=\left(1+\dfrac{1}{2}x+\dfrac{1}{8}x^2-\dots\right)\left(1-2x+3x^2-\dots\right)$ $=1+\left(\dfrac{1}{2}-2\right)x+\left(\dfrac{1}{8}-1+3\right)x^2+\dots$ $\dfrac{\mathrm{e}^{\tan^{-1}\left(\frac{x}{2}\right)}}{\left(1+x\right)^2}=1-\dfrac{3}{2}x+\dfrac{17}{8}x^2+\dots$

5(i) **[2]**	$\overrightarrow{OC}=\frac{2}{3}\mathbf{a}$, $\overrightarrow{OD}=3\mathbf{b}$
(ii) **[4]**	E on BC: $\overrightarrow{OE}=\lambda\overrightarrow{OB}+(1-\lambda)\overrightarrow{OC}$ E on AD: $\overrightarrow{OE}=\mu\overrightarrow{OA}+(1-\mu)\overrightarrow{OD}$ $\overrightarrow{OE}=\lambda\mathbf{b}+(1-\lambda)\cdot\frac{2}{3}\mathbf{a}=\mu\mathbf{a}+(1-\mu)\cdot 3\mathbf{b}$ $\lambda=3-3\mu \quad \cdots(1)$ $2-2\lambda=3\mu \quad \cdots(2)$ (1) + (2): $2-\lambda=3 \Rightarrow \lambda=-1$ $\overrightarrow{OE}=\frac{4}{3}\mathbf{a}-\mathbf{b}$
(iii) **[3]**	Area of CDE is $\frac{1}{2}\left\lvert\overrightarrow{DC}\times\overrightarrow{CE}\right\rvert=\frac{1}{2}\left\lvert\left(\frac{2}{3}\mathbf{a}-3\mathbf{b}\right)\times\left(\frac{2}{3}\mathbf{a}-\mathbf{b}\right)\right\rvert$ $=\frac{1}{2}\left\lvert-\frac{2}{3}\mathbf{a}\times\mathbf{b}+\frac{2}{3}\mathbf{a}\times 3\mathbf{b}\right\rvert$ since $\mathbf{a}\times\mathbf{a}=\mathbf{b}\times\mathbf{b}=\mathbf{0}$ $=\frac{1}{2}\left\lvert\frac{4}{3}\mathbf{a}\times\mathbf{b}\right\rvert=\frac{2}{3}\lvert\mathbf{a}\times\mathbf{b}\rvert$
(iv) **[3]**	$\overrightarrow{OE}=\frac{4}{3}\mathbf{a}-\mathbf{b}$ is the diagonal of the rhombus with sides $\frac{4}{3}\mathbf{a}$ and $-\mathbf{b}$. So $\left\lvert\frac{4}{3}\mathbf{a}\right\rvert=\lvert-\mathbf{b}\rvert \Rightarrow \frac{\lvert\mathbf{a}\rvert}{\lvert\mathbf{b}\rvert}=\frac{3}{4}$, i.e. $OA:OB=3:4$ Alternatively, $\cos\angle AOE=\cos\angle FOE$ $\frac{\mathbf{a}\cdot\left(\frac{4}{3}\mathbf{a}-\mathbf{b}\right)}{\lvert\mathbf{a}\rvert\times OE}=\frac{-\mathbf{b}\cdot\left(\frac{4}{3}\mathbf{a}-\mathbf{b}\right)}{\lvert\mathbf{b}\rvert\times OE}$ $\frac{\frac{4}{3}\lvert\mathbf{a}\rvert^2-\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta}{\lvert\mathbf{a}\rvert}=\frac{-\frac{4}{3}\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta+\lvert\mathbf{b}\rvert^2}{\lvert\mathbf{b}\rvert}$ $\frac{4}{3}\lvert\mathbf{a}\rvert-\lvert\mathbf{b}\rvert\cos\theta=-\frac{4}{3}\lvert\mathbf{a}\rvert\cos\theta+\lvert\mathbf{b}\rvert$ $\frac{4}{3}\lvert\mathbf{a}\rvert(1+\cos\theta)=\lvert\mathbf{b}\rvert(1+\cos\theta)$, $1+\cos\theta\neq 0$ $\frac{\lvert\mathbf{a}\rvert}{\lvert\mathbf{b}\rvert}=\frac{3}{4}$ i.e. $OA:OB=3:4$

Section B: Probability and Statistics [60 marks]

6(i) **[2]**	$\mathrm{P}(X\mid Y')=\dfrac{\mathrm{P}(X\cap Y')}{\mathrm{P}(Y')}$ $\mathrm{P}(Y')=\dfrac{\mathrm{P}(X\cap Y')}{\mathrm{P}(X\mid Y')}$ $=\dfrac{\left(1/2\right)}{\left(50/63\right)}$ $=\dfrac{63}{100}$
(ii) **[2]**	$\mathrm{P}(X\cup Y')=\mathrm{P}(X)+\mathrm{P}(Y')-\mathrm{P}(X\cap Y')$ $\mathrm{P}(X)=\mathrm{P}(X\cup Y')+\mathrm{P}(X\cap Y')-\mathrm{P}(Y')$ $=\dfrac{3}{4}+\dfrac{1}{2}-\dfrac{63}{100}$ $\mathrm{P}(X)=\dfrac{31}{50}$
(iii) **[3]**	$\mathrm{P}(X)=\mathrm{P}(X\cap Y)+\mathrm{P}(X\cap Y')$ $\mathrm{P}(X\cap Y)=\mathrm{P}(X)-\mathrm{P}(X\cap Y')$ $=\dfrac{31}{50}-\dfrac{1}{2}$ $=\dfrac{3}{25}$ Since $\mathrm{P}(X\cap Y)=\dfrac{3}{25}\neq\dfrac{31}{50}\times\dfrac{37}{100}=\mathrm{P}(X)\times\mathrm{P}(Y)$, X and Y are not independent events OR Since $\mathrm{P}(X)=\dfrac{31}{50}\neq\dfrac{50}{63}=\mathrm{P}(X\mid Y')$, X and Y' are not independent events So X and Y are not independent events

7(i) **[5]**	Let X be the mass of a bag of sugar in kg. The necessary assumption is X follows a normal distribution. Assumption that X follows a normal distribution is required as question did not state the distribution of X and n = 8 is too small to use Central Limit Theorem. $H_0: \quad \mu = 1$ $H_1: \quad \mu > 1$ Under H_0, $\bar{X} \sim N\left(1, \frac{0.08^2}{8}\right)$ $\bar{x} = \frac{8.4}{8} = 1.05$ Test Statistic: $Z = \frac{\bar{X} - 1}{0.08/\sqrt{8}}$ Level of significance: 5% Reject H_0 if p-value < 0.05 Using GC, p-value $= 0.0385 < 0.05$ Since p-value = 0.0385 < 0.05, we reject H_0 and conclude there is sufficient evidence, at the 5% significance level, to support the manufacturer's concern.
(ii) **[3]**	Under H_0, $\bar{X} \sim N\left(1, \frac{\sigma^2}{8}\right)$ H_0 not rejected $\Rightarrow$ p-value > 0.05 $P\left(\bar{X} > 1.05\right) > 0.05$ $P\left(Z > \frac{1.05 - 1}{\sqrt{\frac{\sigma^2}{8}}}\right) > 0.05$ $0.05\sqrt{\frac{8}{\sigma^2}} < 1.64485$ $\sigma^2 > 8\left(\frac{0.05}{1.64485}\right)^2$ $\sigma^2 > 0.007392$

8 (i) [3]	Let X and Y be the masses in grams of a randomly chosen apple and a randomly chosen potato in grams respectively. i.e. $X \sim \mathrm{N}(90, 13^2)$, $Y \sim \mathrm{N}(170, 30^2)$. $\mathrm{E}\left(Y - 2X\right) = 170 - 2(90) = -10$ $\mathrm{Var}\left(Y - 2X\right) = 30^2 + 2^2(13^2) = 1576$ $Y - 2X \sim \mathrm{N}\left(-10, 1576\right)$ Required probability $= \mathrm{P}(Y > 2X)$ $= \mathrm{P}(Y - 2X > 0)$ $= 0.401$ (3 s.f.)
(ii) [3]	Let $T = X_1 + X_2 + ... + X_5 + Y_1 + Y_2 + ... + Y_6$. $\mathrm{E}\left(T\right) = 5(90) + 6(170) = 1470$ $\mathrm{Var}\left(T\right) = 5(13^2) + 6(30^2) = 6245$ $T \sim \mathrm{N}\left(1470, 6245\right)$ Required probability $= \mathrm{P}(1200 < T < 1500)$ $= 0.648$ (3 s.f.)
(iii) [3]	Let $W = 0.85\left(X_1 + X_2 + ... + X_5\right) + 0.75\left(Y_1 + Y_2 + ... + Y_6\right)$ $\mathrm{E}\left(W\right) = (0.85)(5)(90) + (0.75)(6)(170) = 1147.5$ $\mathrm{Var}\left(W\right) = (0.85^2)(5)(13^2) + (0.75^2)(6)(30^2) = 3648.0125$ $W \sim \mathrm{N}\left(1147.5,\ 3648.0125\right)$ Required probability $= \mathrm{P}(W \le 1200)$ $= 0.808$ (3 s.f.)

9 (i) [2]	$\mathrm{P}(k < X < 7) = \mathrm{P}(X < 7) - \mathrm{P}(X < k) = 0.8 - 0.2 = 0.6$ (shown) $\mathrm{P}(\mu < X < 7) = 0.3$
(ii) [1]	Since $\mathrm{P}(k < X < \mu) = \mathrm{P}(\mu < X < 7) = 0.3,$ by symmetry $\mu = \frac{k+7}{2}.$
(iii) [2]	$\mathrm{P}(X < 7) = 0.8 \Rightarrow \mathrm{P}(Z < \frac{7-\mu}{\sqrt{12}}) = 0.8$ Therefore $\frac{7-\mu}{\sqrt{12}} = 0.8416$ (4 d.p) $\frac{7-\mu}{\sqrt{12}} = 0.8416 \Rightarrow \mu = 4.0845$
(iv) [3]	$k = 2\mu - 7 = 1.169$ $\mathrm{P}(\lvert X \rvert < k) = 0.135$ (3 s.f)
(v) [4]	$2\mathrm{P}(X < r) = 3\mathrm{P}(X > r)$ $\Rightarrow \mathrm{P}(X > r) = 0.4$ Let Y be the number of observations out of 10 with values greater than r. $Y \sim \mathrm{B}(10, 0.4)$ $\mathrm{P}(Y \geq 6) = 1 - \mathrm{P}(Y \leq 5) = 0.166$ (3s.f)

10(i) [1]	number of committees if there is no restrictions in the selection $= {}^{13}C_7 = 1716$
(i) [2]	There are 3 cases: Case 1: 4 women 3 men Number of committees = ${}^{6}C_4 \times {}^{7}C_3 = 525$ Case 2: 5 women 2 men Number of committees = ${}^{6}C_5 \times {}^{7}C_2 = 126$ Case 3: 6 women 1 man Number of committees = ${}^{6}C_6 \times {}^{7}C_1 = 7$ Total number of committees = 658 (Shown)
(iii) [1]	P(the committee will consist of 4 single women and 3 single men) $= \frac{{}^{5}C_4 \times {}^{6}C_3}{658} = \frac{100}{658} = \frac{50}{329}$
(iv) [2]	Number of committees with no married member $= {}^{5}C_4 \times {}^{6}C_3 + {}^{5}C_5 \times {}^{6}C_2$ $= 100 + 15$ $= 115$

	Number of committees with at least one married member $= 658 - 115$ $= 543$ P(the committee will contain at least one married member) $= \dfrac{543}{658}$ **<u>Alternative Solution</u>** We consider 3 cases: Case 1: Husband in, wife out Number of committees $= {}^5C_4 \times {}^6C_2 + {}^5C_5 \times {}^6C_1 = 75 + 6 = 81$ Case 2: Wife in, husband out Number of committees $= {}^5C_3 \times {}^6C_3 + {}^5C_4 \times {}^6C_2 + {}^5C_5 \times {}^6C_1 = 200 + 75 + 6 = 281$ Case 3: Both husband and wife are in Number of committees $= {}^5C_3 \times {}^6C_2 + {}^5C_4 \times {}^6C_1 + {}^5C_5 = 150 + 30 + 1 = 181$ Total number of such committees = 543 P(the committee will contain at least one married member) $= \dfrac{543}{658}$
(v) **[2]**	Number of ways to sit 4 women around the table = 3! Number of ways to slot the 3 men = $4 \times 3 \times 2 = 24$ Number of sitting arrangements where the all the men are separated $= 3! \times 24$ $= 144$ Required probability $= \dfrac{144}{6!}$ $= \dfrac{1}{5}$
(vi) **[3]**	Number of ways to sit 4 women around the table = 3! Number of ways to sit the man next to his wife = 2 Number of ways to sit the other 2 men = ${}^3P_2$ Number of ways to have all the men separated from each other and the married couple sit next to each other $= 3! \times 2 \times 3 \times 2 = 72$ Required probability $= \dfrac{72}{144}$ $= \dfrac{1}{2}$

11 (i) [2]	<table><tr><td>x</td><td>2</td><td>5</td><td>6</td><td>8</td><td>9</td><td>10</td></tr><tr><td>$\mathrm{P}(X=x)$</td><td>$\frac{1}{4}$ =0.25</td><td>$\frac{3}{10}$ =0.3</td><td>$\frac{1}{5}$ =0.2</td><td>$\frac{9}{100}$ =0.09</td><td>$\frac{3}{25}$ =0.12</td><td>$\frac{1}{25}$ =0.04</td></tr></table>
(ii) [3]	Using GC, E(X) = 5.4 $\mathrm{E}\left(X^2\right)=35.18$ $\mathrm{Var}\left(X\right)=\mathrm{E}\left(X^2\right)-\left[\mathrm{E}\left(X\right)\right]^2=35.18-(5.4)^2=6.02$ (shown)
(iii) [3]	Since $n=50$ is large, by Central Limit Theorem, $\overline{X}\sim \mathrm{N}\left(5.4,\frac{6.02}{50}\right)$ approximately . Required probability $=\mathrm{P}\left(\overline{X}\geq 6\right)=0.0419$ (3 s.f.)
(iv) [4]	$\mathrm{P}\left(\text{winning a cash voucher}\right)$ $=\mathrm{P}\left(X>6\right)$ $=\mathrm{P}\left(X=8,9\text{ or }10\right)$ $=\frac{9}{100}+\frac{3}{25}+\frac{1}{25}=\frac{1}{4}$ $\therefore Y\sim \mathrm{B}\left(n,\frac{1}{4}\right)$ We must have $\mathrm{P}\left(Y>3\right)>0.7$ $\Rightarrow 1-\mathrm{P}\left(Y\leq 3\right)>0.7$ $\Rightarrow \mathrm{P}\left(Y\leq 3\right)<0.3$ Using GC, <table><tr><td>n</td><td>$\mathrm{P}\left(Y\leq 3\right)$</td></tr><tr><td>18</td><td>0.30569</td></tr><tr><td>19</td><td>0.26309</td></tr></table>$\therefore$ Least $n=19$